Easy to Win, Hard to Master:
Optimal Strategies in Parity Games with Costs

Joint work with Martin Zimmermann

Alexander Weinert

Saarland University

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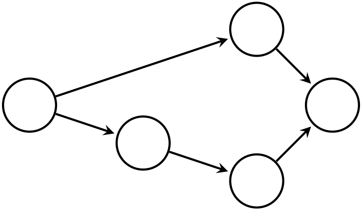
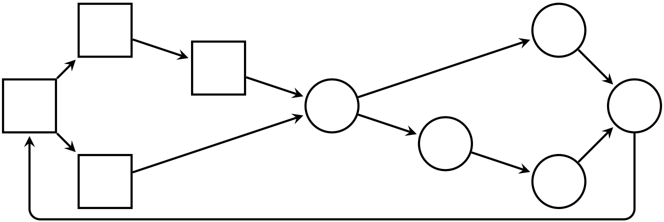
MFV Seminar, ULB, Brussels, Belgium

Parity Games



Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Parity Games



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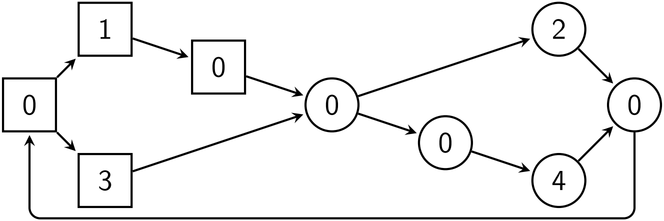

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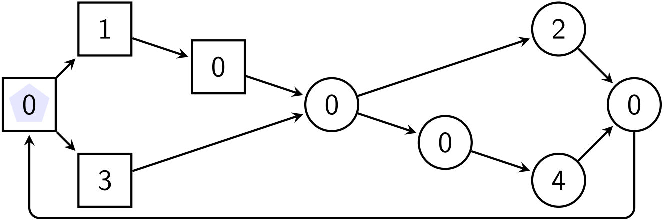

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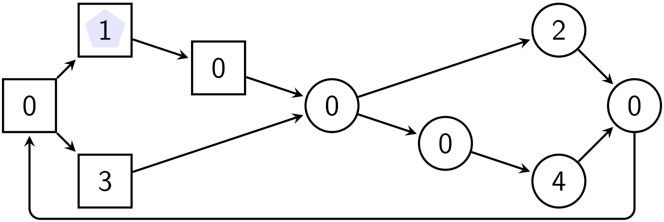
Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

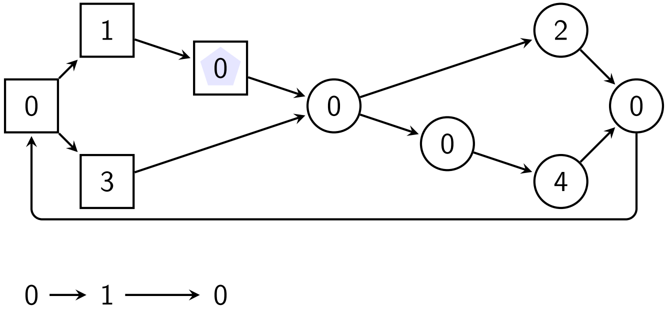
Parity Games



$0 \longrightarrow 1$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Parity Games



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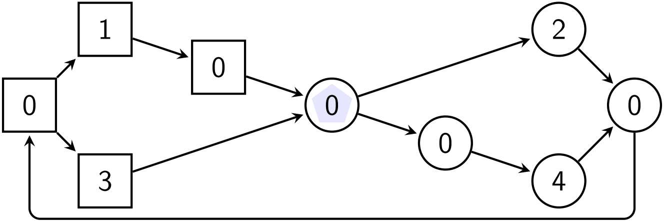
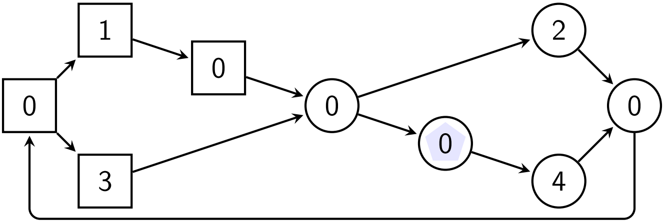


$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Parity Games



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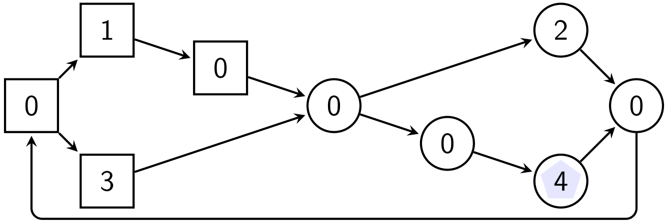
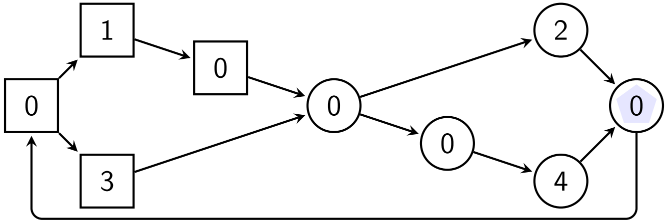


$0 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 4$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Parity Games



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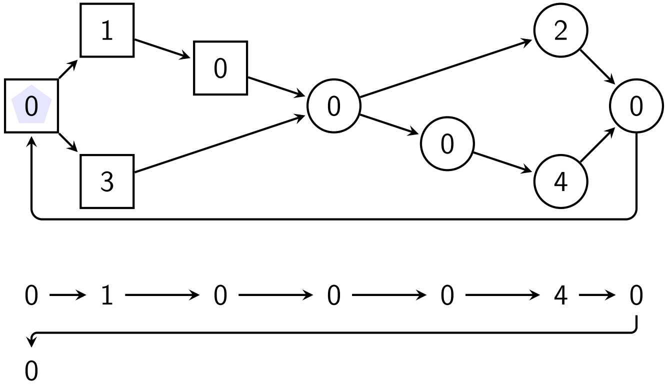

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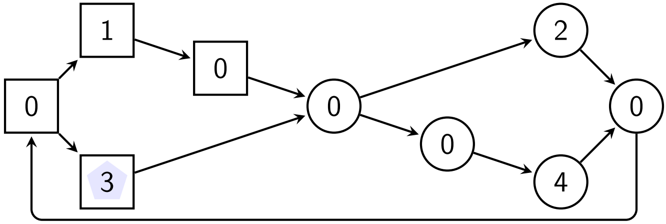
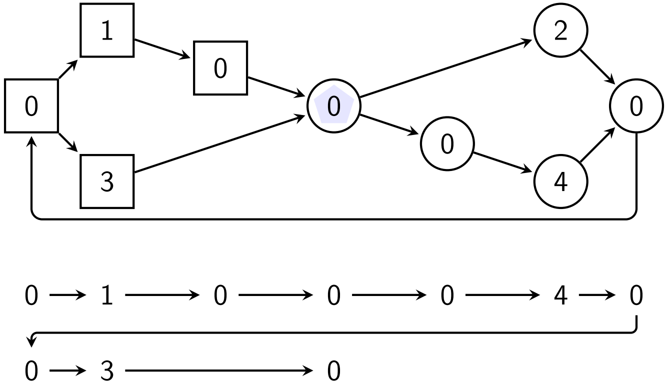


$0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0$

$0 \rightarrow 3$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Parity Games



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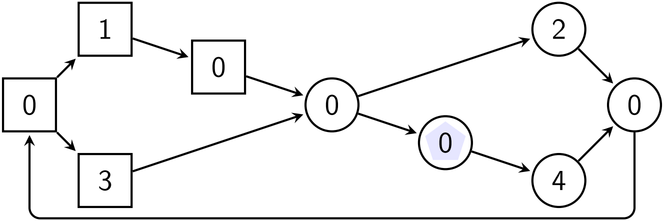

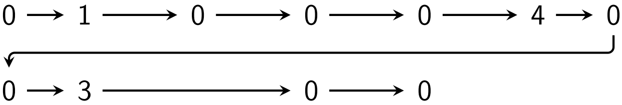

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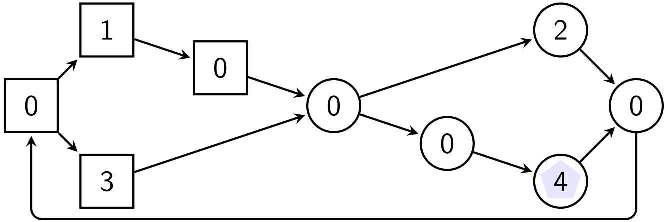
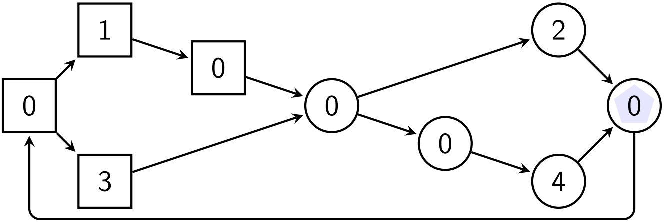


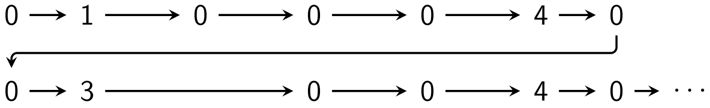
$0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0$

$0 \rightarrow 3 \rightarrow 0 \rightarrow 0 \rightarrow 4$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Parity Games





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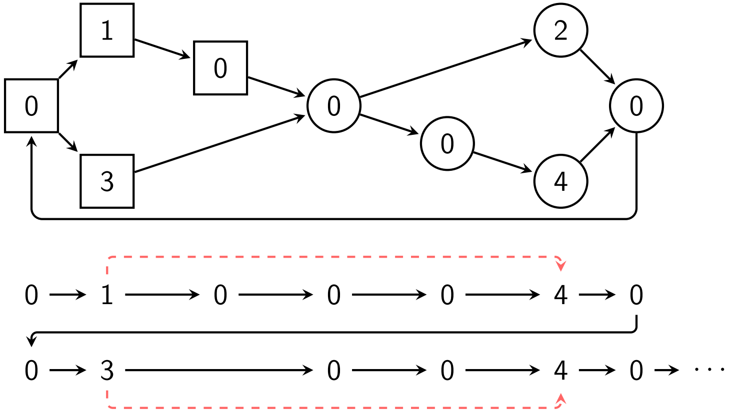

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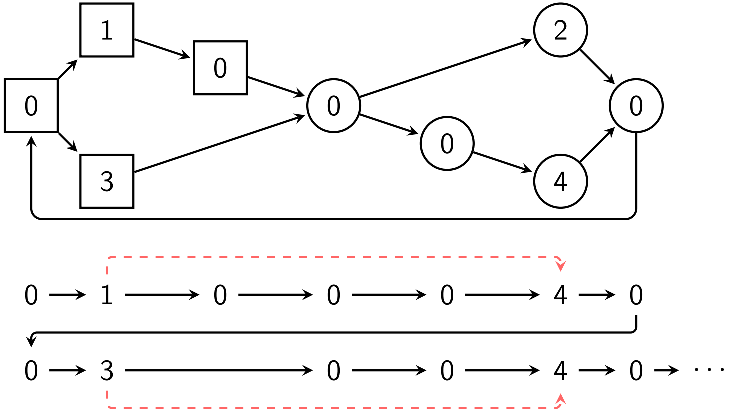
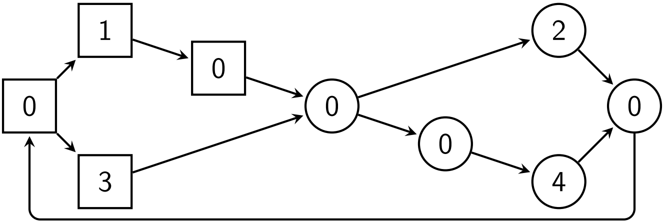


Deciding winner in $\mathrm{UP} \cap \mathrm{CO\text{-}UP}$

Positional Strategies

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Finitary Parity / Parity Response Games



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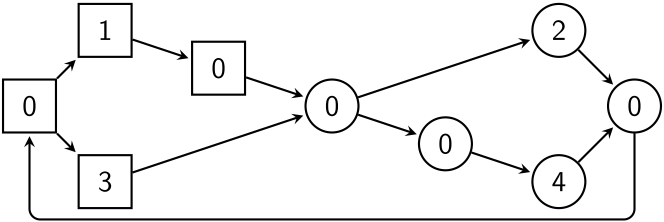

$0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0$

$0 \rightarrow 3 \rightarrow 0 \rightarrow 0 \rightarrow 4 \rightarrow 0 \rightarrow \cdots$

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

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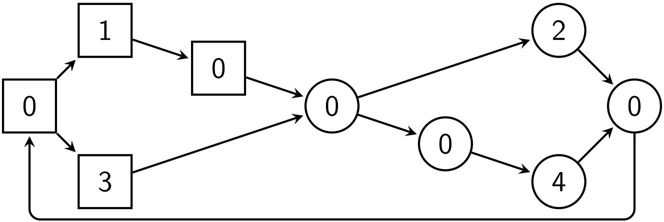

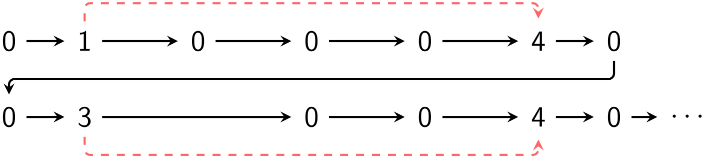

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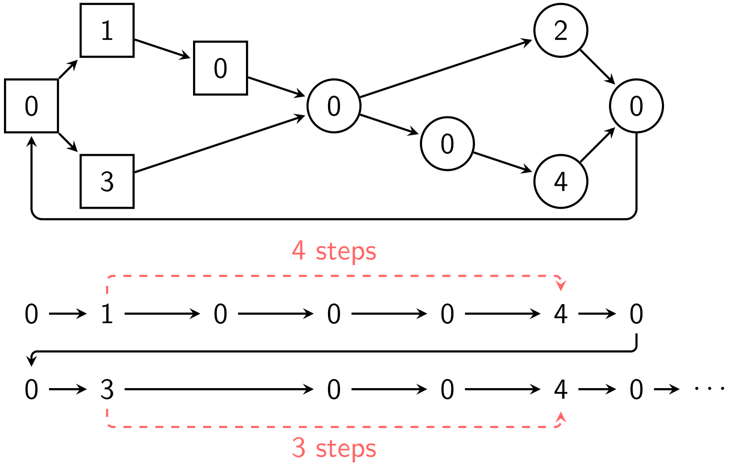

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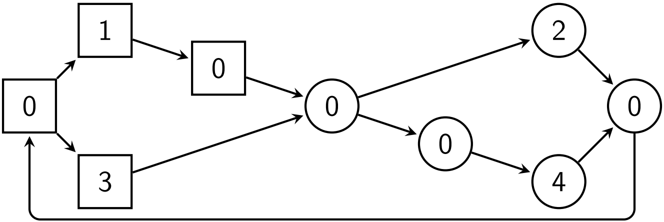
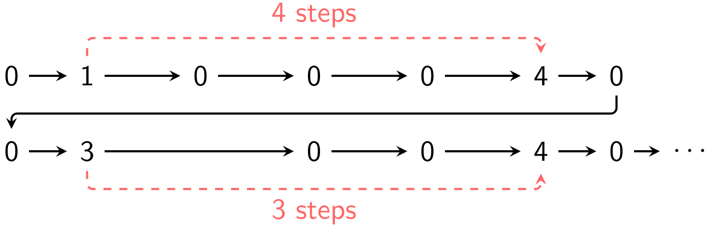
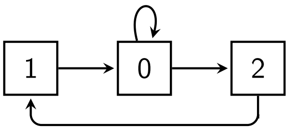




Goal for Player 0: Bound response times

Example due to (Fijalkow and Chatterjee, Infinite-state games, 2013)

Another Example



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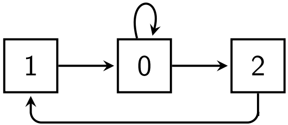

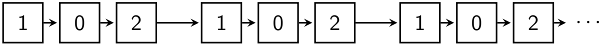

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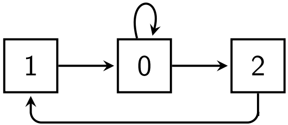
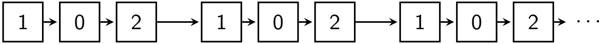
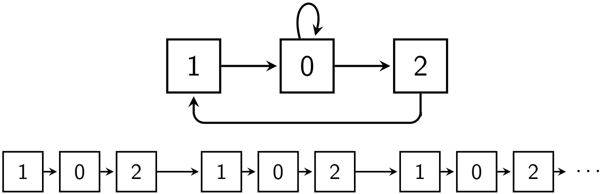




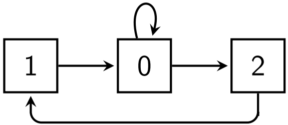
Parity ✓

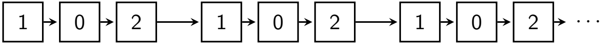
Another Example



Parity ✓ Finitary Parity ✓

Another Example





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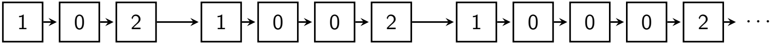

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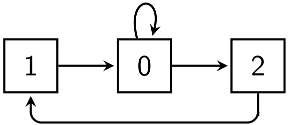

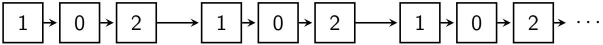

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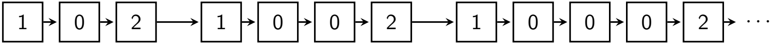

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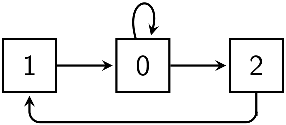

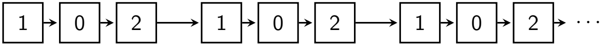

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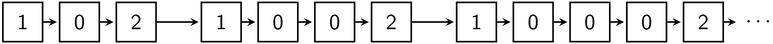

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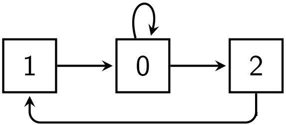

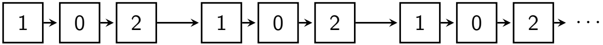

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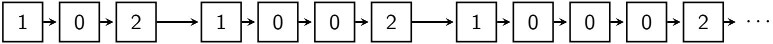

Parity ✓ Finitary Parity ✗

- Player 1 wins from every vertex,
- but needs to stay longer and longer in vertex of color 0
  $\Rightarrow$ requires infinite memory

Decision Problem

Theorem (Chatterjee et al., Finitary Winning, 2009)

The following decision problem is in $\textsc{PTime}$*:*

Input: *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$

Question: *Does there exist a strategy* σ *with* $\mathrm{Cst}(\sigma) < \infty$*?*

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Theorem

The following decision problem is $\textsc{PSpace}$*-complete:*

Input: *Finitary parity game* $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$*, bound* $b \in \mathbb{N}$

Question: *Does there exist a strategy* σ *with* $\mathrm{Cst}(\sigma) \leq b$*?*

Introduction

Introduction ✓

Introduction ✓

↓

Complexity

in PSpace

Introduction ✓

↓

Complexity

in PSpace

PSpace-hard

Introduction ✓

↓

Complexity

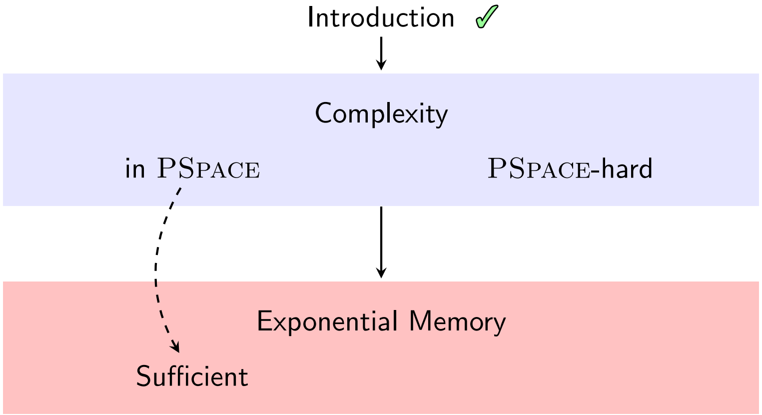
in PSpace

PSpace-hard

↓

Exponential Memory

Sufficient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in PSPACE
PSPACE-hard
Exponential Memory
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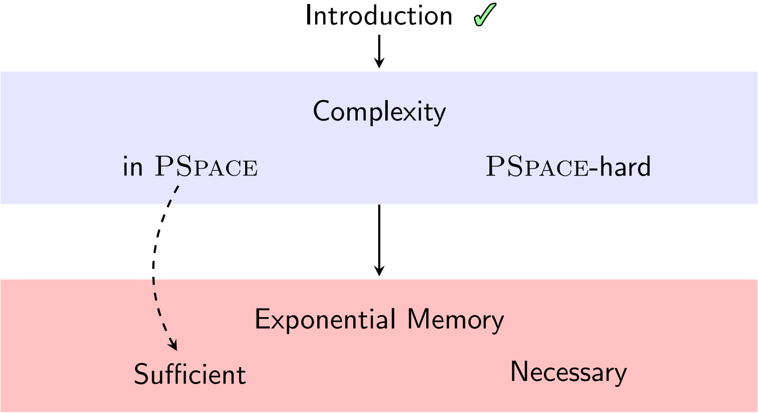
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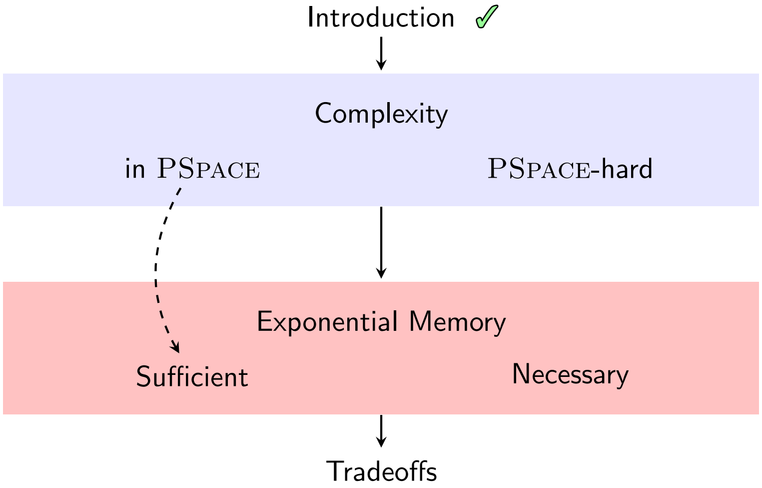
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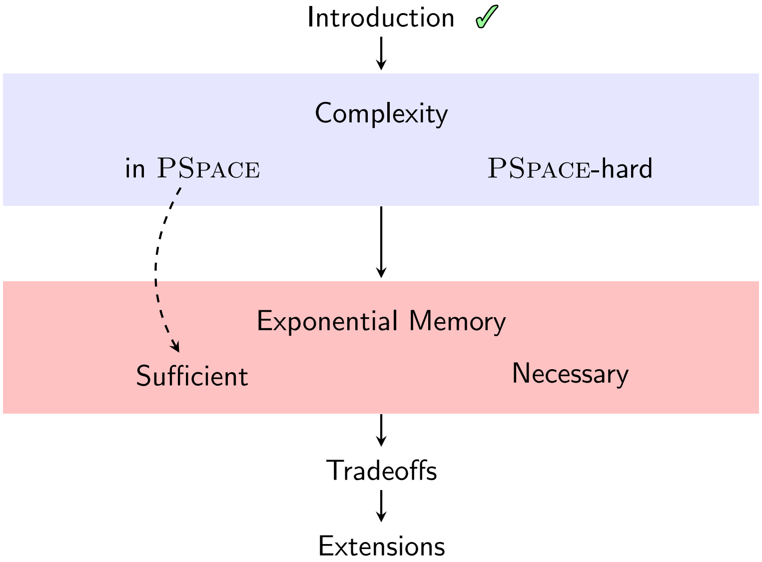
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Extensions

From Finitary Parity to Parity

Given: Finitary parity game $\mathcal{G} = (\mathcal{A}, \mathrm{FinParity}(\Omega))$, bound $b \in \mathbb{N}$.

Lemma

Deciding if Player 0 has strategy σ with $\mathrm{Cst}(\sigma) \leq b$ is in $\mathrm{PS{\scriptsize PACE}}$.

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Idea: Simulate $\mathcal{G}$, keeping track of open requests explicitly.

Result: Parity game $\mathcal{G}'$ of exponential size.

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(Chandra et al., Alternation, 1981)

$\Rightarrow$ Problem is in PSPACE

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Complexity

in PSpace

PSpace-hard

Exponential Memory

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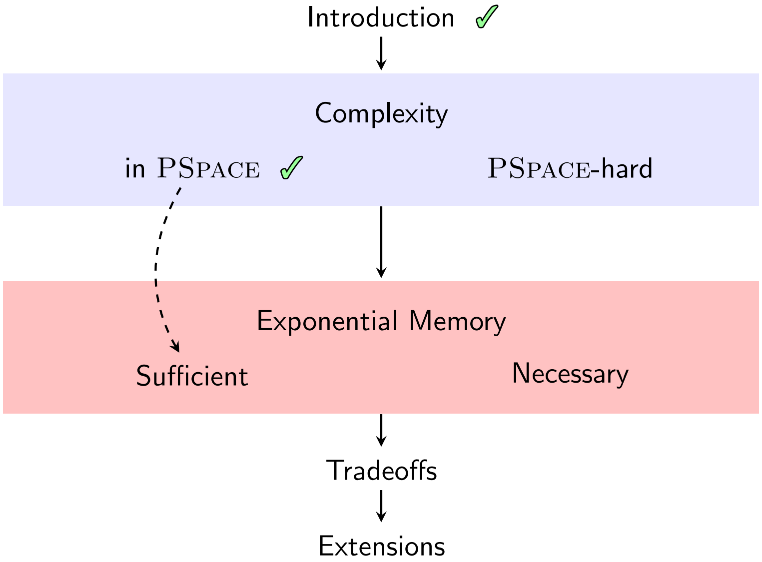
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PSPACE-Hardness

Lemma

The following problem is $\mathrm{PS}\textsc{pace}$*-hard: "Given a finitary parity game* $\mathcal{G}$ *and a bound* $b \in \mathbb{N}$*, does Player* 0 *have a strategy* σ *for* $\mathcal{G}$ *with* $\mathrm{Cst}(\sigma) \leq b$*?"*

PSPACE-Hardness

Lemma

The following problem is $\text{PS}\textsc{pace}$*-hard: "Given a finitary parity game* $\mathcal{G}$ *and a bound* $b \in \mathbb{N}$*, does Player* 0 *have a strategy* σ *for* $\mathcal{G}$ *with* $\text{Cst}(\sigma) \leq b$*?"*

Proof

- By reduction from QBF
- Checking the truth of $\varphi = \forall x \exists y.\ (x \lor \neg y) \land (\neg x \lor y)$ as a two-player game (Player 0 wants to prove truth of φ):

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  1. Player 1 picks truth value for x
  2. Player 0 picks truth value for y
  3. Player 1 picks clause C
  4. Player 0 picks literal ℓ from C
  5. Player 0 wins $\Leftrightarrow$ ℓ is picked to be satisfied in step 1 or 2

The Reduction

$$\varphi = \forall x \ \exists y \ . \ \overbrace{(\, x \vee \neg y \,) \wedge (\, \neg x \vee y \,)}^{\psi}$$

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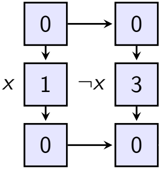

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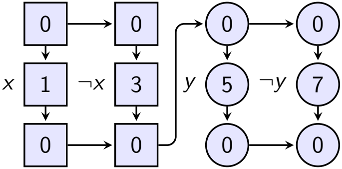

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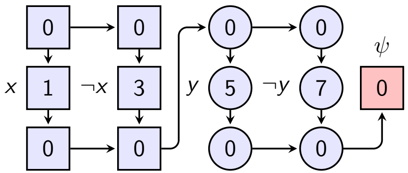

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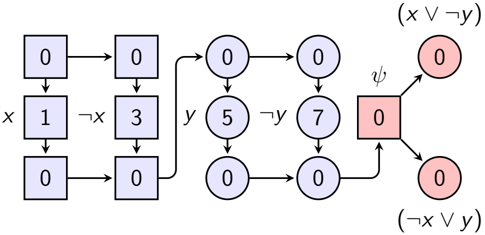

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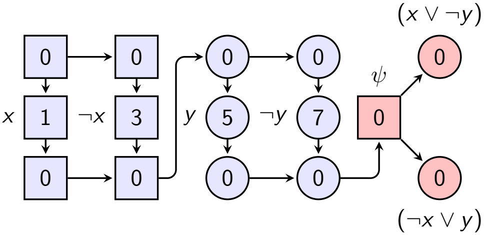

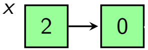

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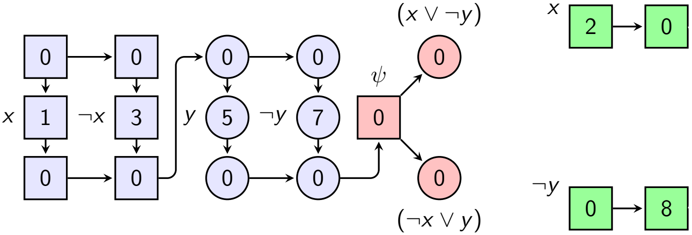

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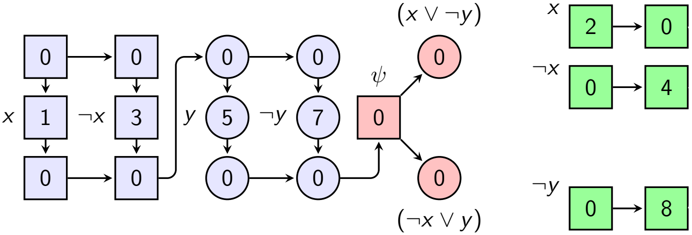

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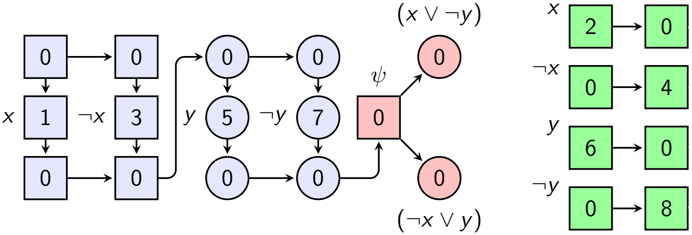

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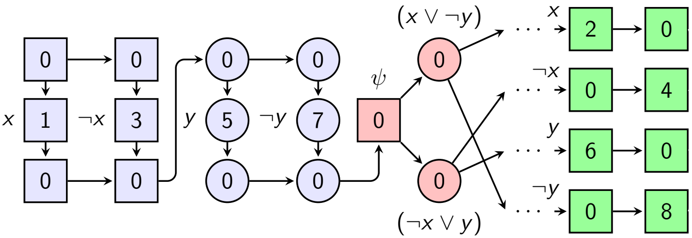

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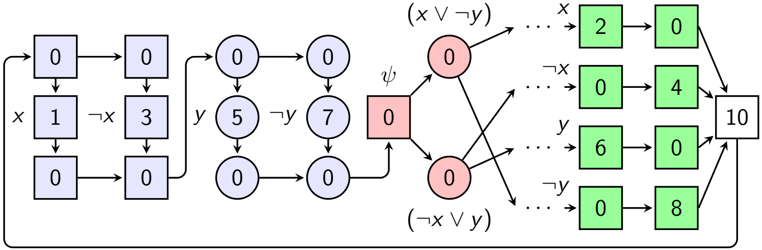

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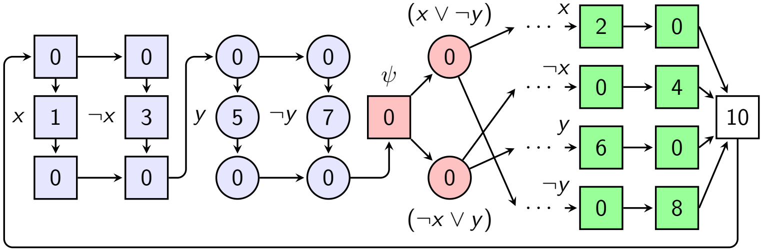

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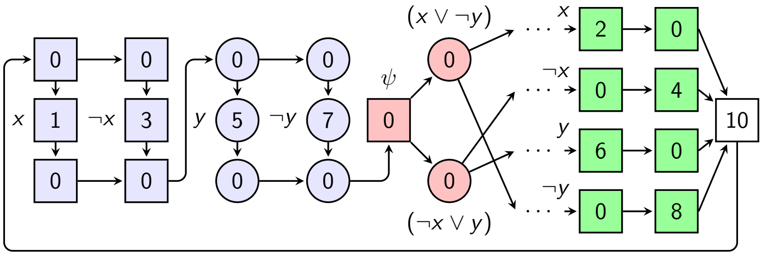

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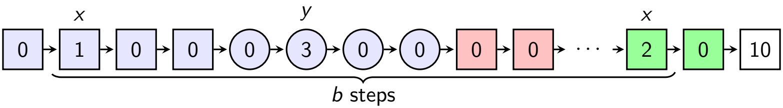

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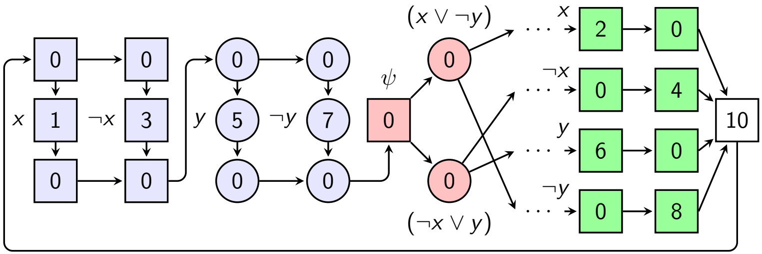

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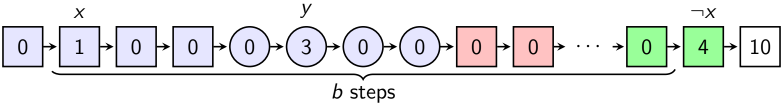

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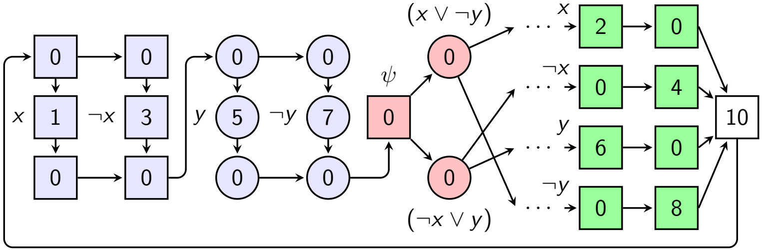

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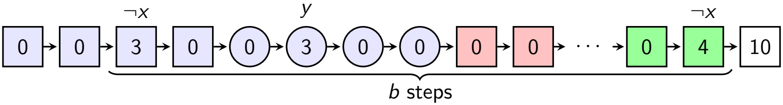

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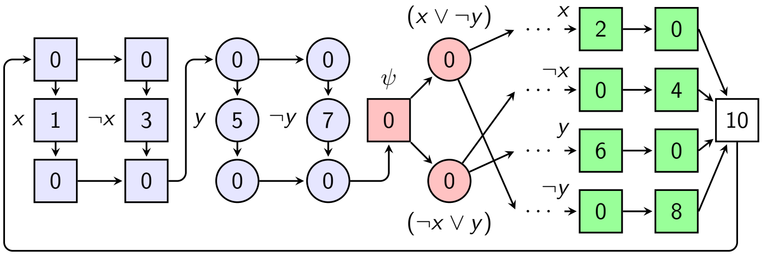

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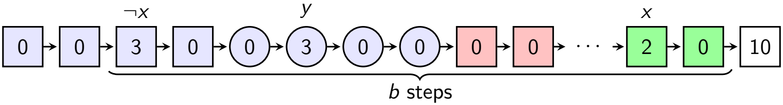

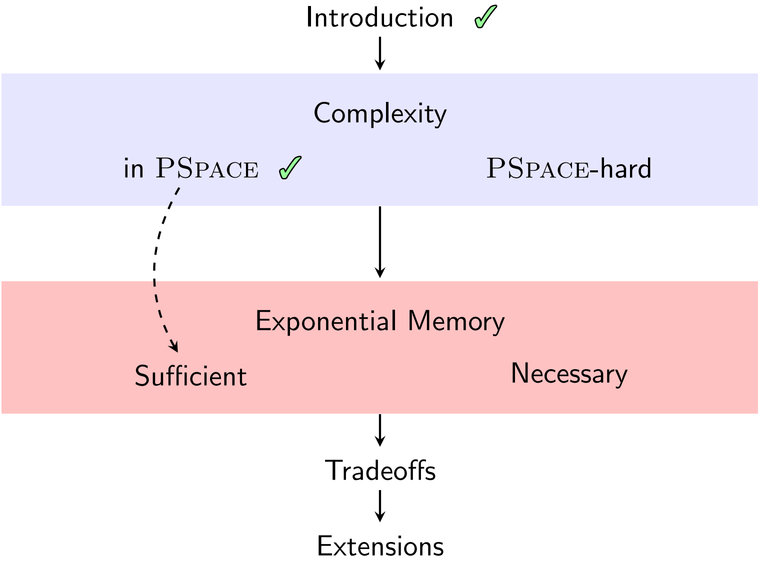
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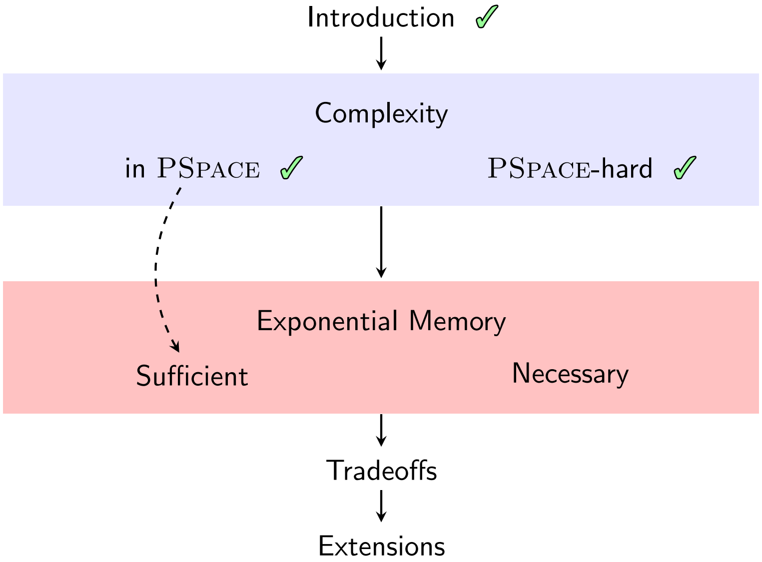
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Sufficient Memory (for Player 0)

Corollary

Let $\mathcal{G}$ be a parity game with costs with d odd colors.
If Player 0 has a strategy σ for $\mathcal{G}$ with $\mathrm{Cst}(\sigma) = b$, then she also has a strategy σ' with $\mathrm{Cst}(\sigma') = b$ and size $(b+2)^d = 2^{d \log(b+2)}$.

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Follows from

- proof of PSPACE-membership and
- positional strategies for parity games.

Memory Requirements (for Player 0)

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Optimal strategies for parity games require exponential memory.

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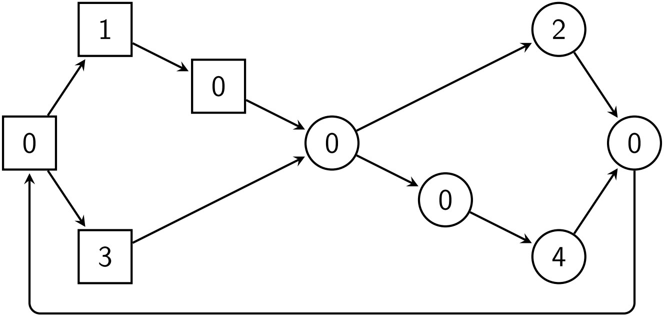

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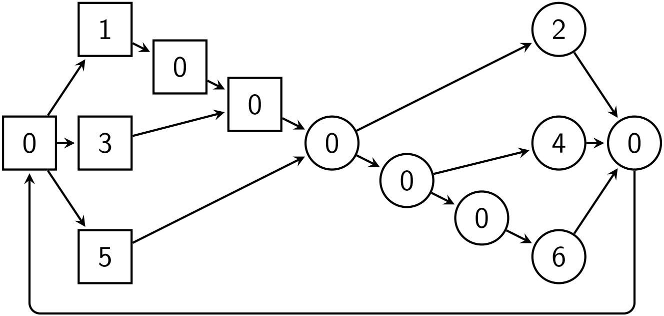

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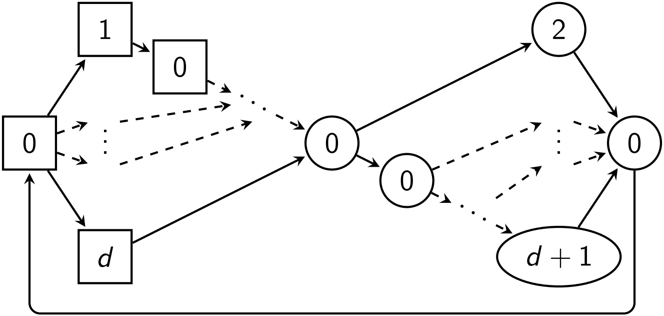

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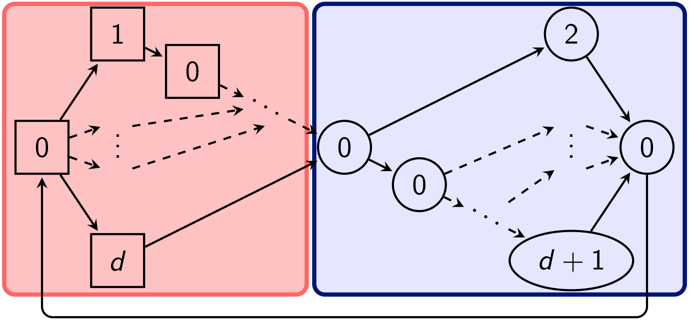

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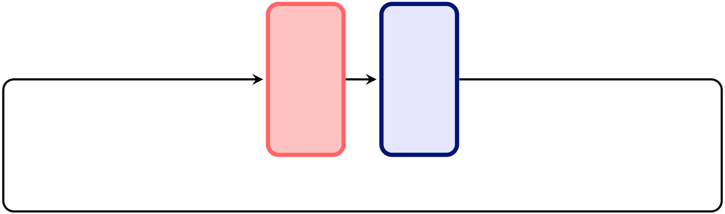

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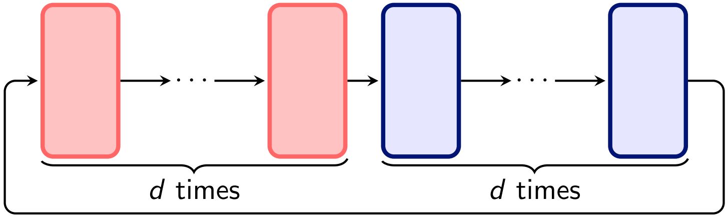

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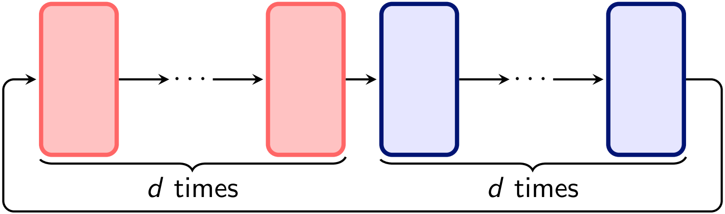

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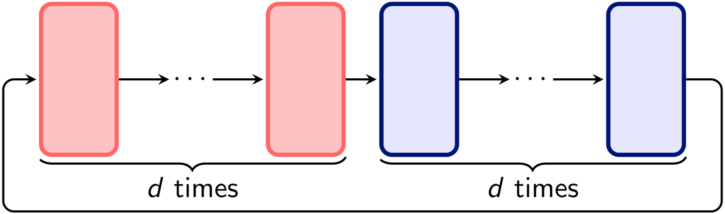

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$\Rightarrow$ Player 0 requires $\approx 2^d$ many memory states

Memory Requirements (cont.)

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For every $d > 1$, there exists a finitary parity game $\mathcal{G}_d$ such that

- $|\mathcal{G}_d| \in \mathcal{O}(d^2)$ *and* $\mathcal{G}_d$ *has* d *odd colors, and*
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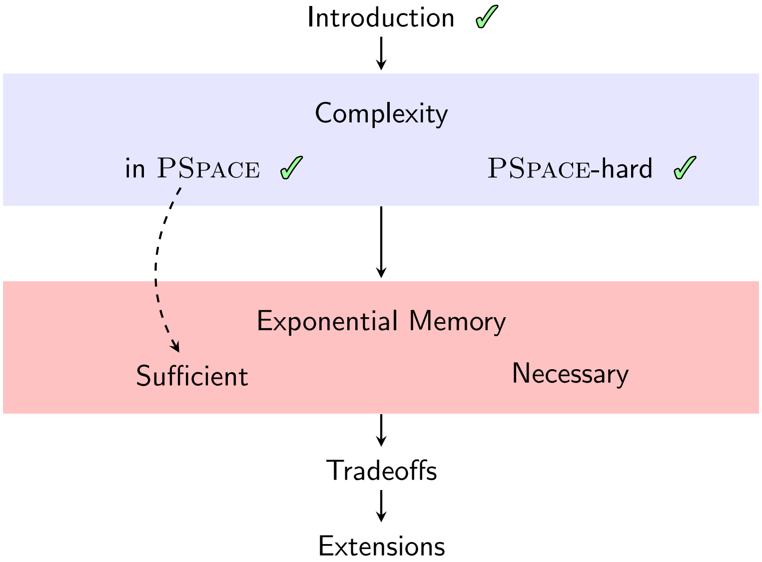
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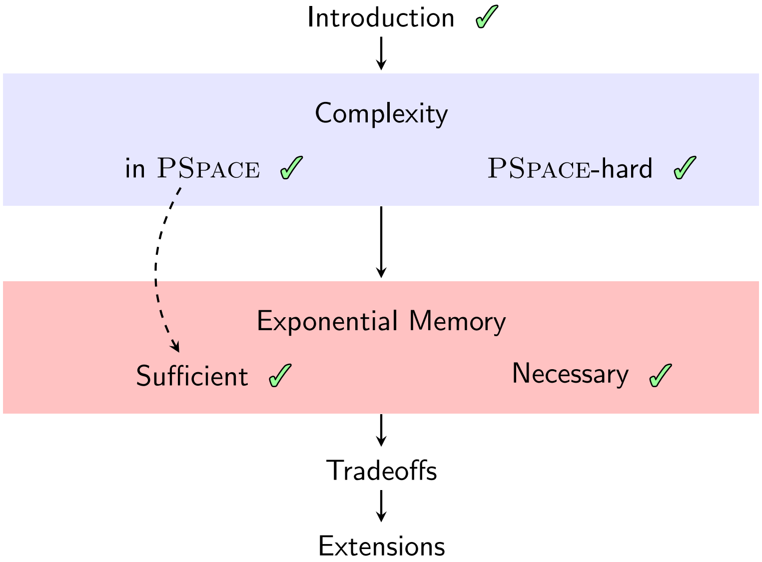
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Results so far

	Parity		
Complexity	$\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$		
Strategies	Pos.		

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	Parity	Finitary Parity	
		Winning	
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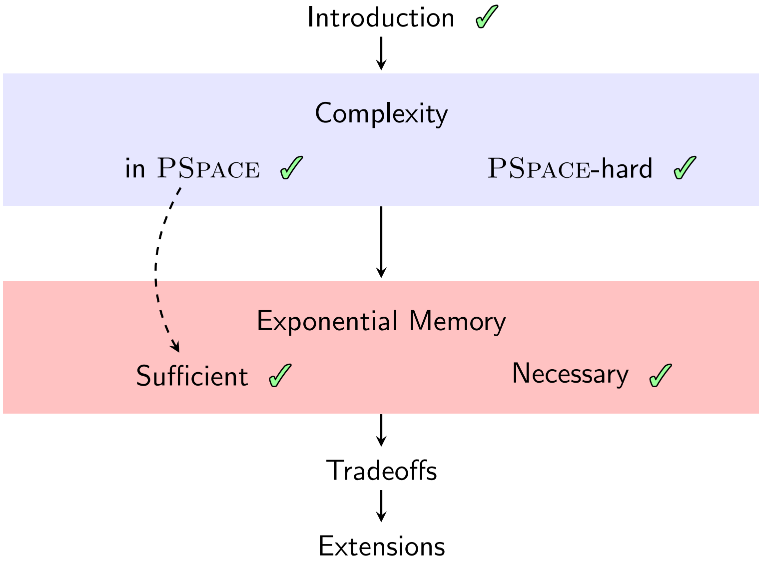
Take-away: Forcing Player 0 to answer quickly in (finitary) parity games makes it harder

- to decide whether she can satisfy the bound
- for her to play the game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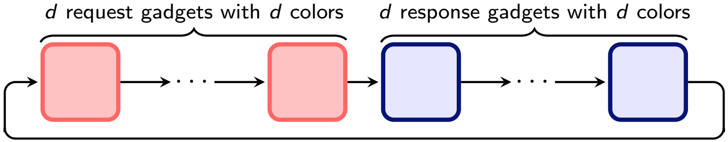
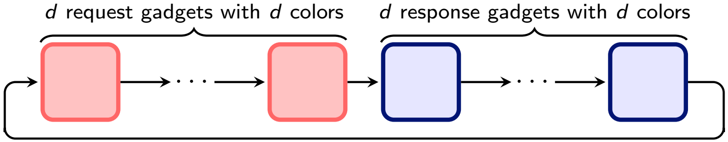
Tradeoffs

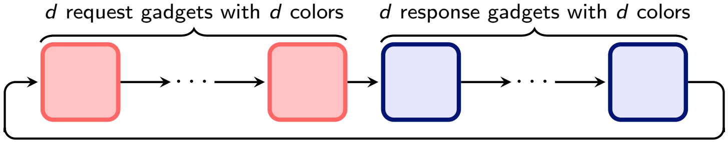

d request gadgets with d colors
d response gadgets with d colors

Tradeoffs



- **Recall:** Player 0 has winning strategy with cost $d^2 + 2d$ and size $2^d - 2$: store all d requests made by Player 1.

Tradeoffs



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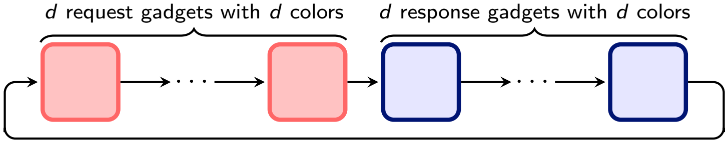

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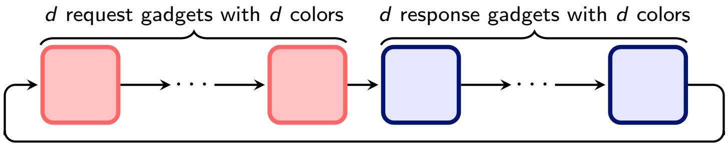

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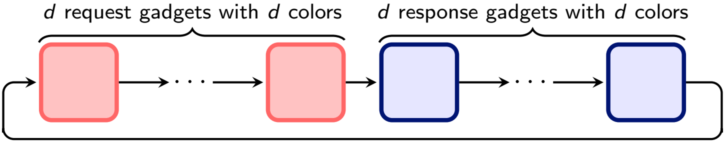

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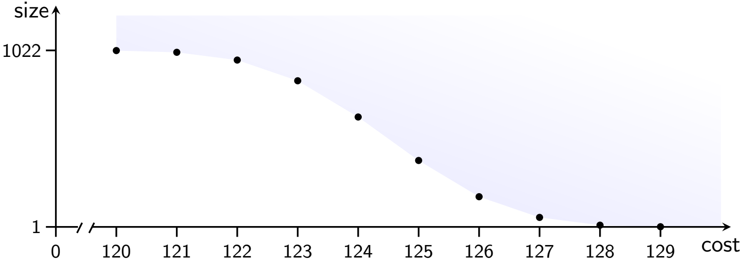
- These are the smallest strategies achieving this cost.

Tradeoffs

Theorem

Fix some finitary parity game $\mathcal{G}_d$ as before. For every i with $1 \leq i \leq d$ there exists a strategy σ_i for Player 0 in $\mathcal{G}_d$ such that σ_i has cost $d^2 + 3d - i$ and size $\sum_{j=1}^{i-1} \binom{d}{j}$.
Also, all these strategies are minimal for their respective cost.



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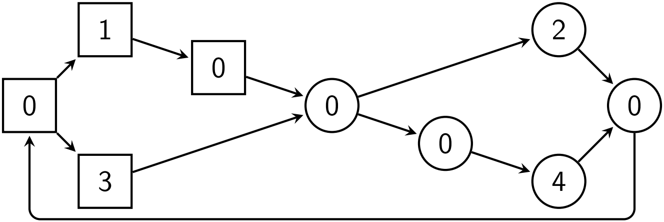
Extension 1: Parity Games with Costs



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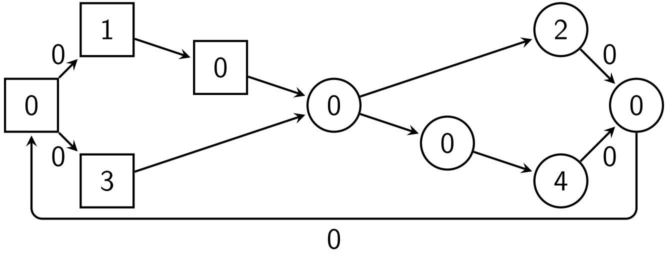

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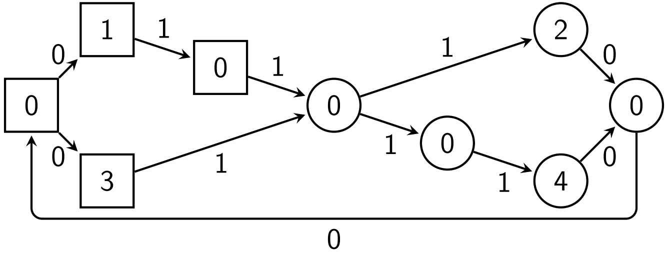
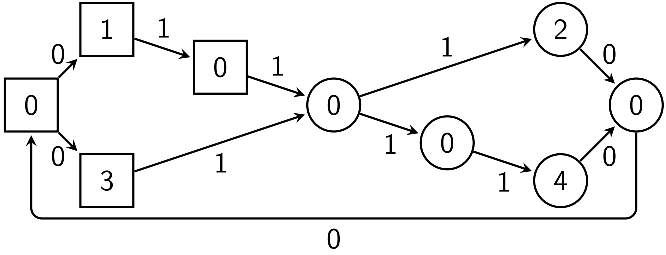


Extension 1: Parity Games with Costs



Finitary parity games are special case

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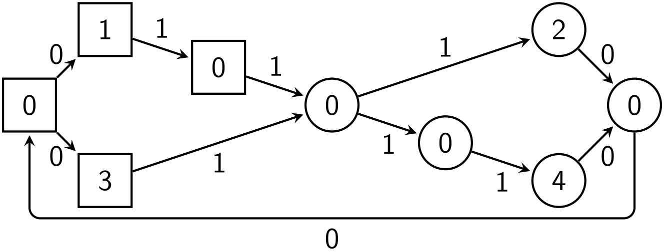

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⇒ PSpace-hard ⇒ Exp. memory necessary

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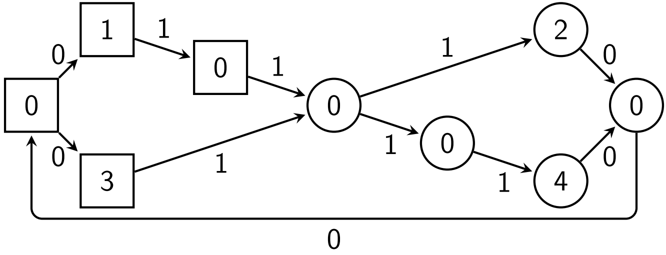
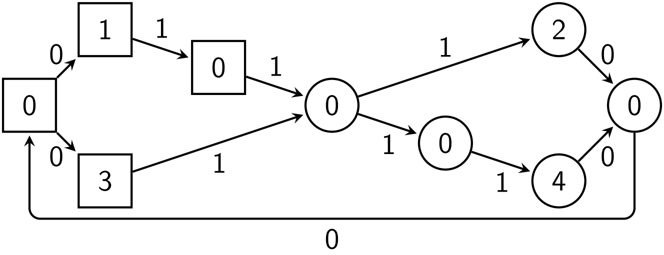


Finitary parity games are special case

$\Rightarrow$ PSPACE-hard $\Rightarrow$ Exp. memory necessary

Algorithm for finitary games works with some extensions

Extension 1: Parity Games with Costs



Finitary parity games are special case

⇒ PSpace-hard ⇒ Exp. memory necessary

Algorithm for finitary games works with some extensions

⇒ In PSpace ⇒ Exp. memory sufficient

Extension 2: Streett

Finitary Streett Games

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No jump in complexity:

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Streett Games with Costs

- Deciding winner $\textsc{ExpTime}$-complete
- Exponential memory necessary and sufficient

Conclusion

	Parity
Complexity	$\mathrm{UP} \cap \mathrm{co\text{-}UP}$
Strategies	Pos.

Conclusion

	Parity	Parity with Costs
		Winning
Complexity	$\mathrm{UP} \cap \mathrm{co\text{-}UP}$	$\mathrm{UP} \cap \mathrm{co\text{-}UP}$
Strategies	Pos.	Pos.

Conclusion

	Parity	Parity with Costs	
		Winning	Optimal
Complexity	$\mathrm{UP} \cap \mathrm{co\text{-}UP}$	$\mathrm{UP} \cap \mathrm{co\text{-}UP}$	PSPACE-comp.
Strategies	Pos.	Pos.	Exp.

Conclusion

	Parity	Parity with Costs	
		Winning	Optimal
Complexity Strategies	$\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$ Pos.	$\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$ Pos.	PSPACE-comp. Exp.

	Streett		
Complexity Strategies	co-NP Exp.		

Conclusion

	Parity	Parity with Costs	
		Winning	Optimal
Complexity	UP ∩ co-UP	UP ∩ co-UP	PSpace-comp.
Strategies	Pos.	Pos.	Exp.

	Streett	Streett with Costs
		Winning
Complexity	co-NP	ExpTime
Strategies	Exp.	Exp.

Conclusion

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Slides available at `react.uni-saarland.de/people/weinert.html`