Optimal Strategy Synthesis for Request-Response Games

Joint work with Florian Horn, Nico Wallmeier, and Wolfgang Thomas

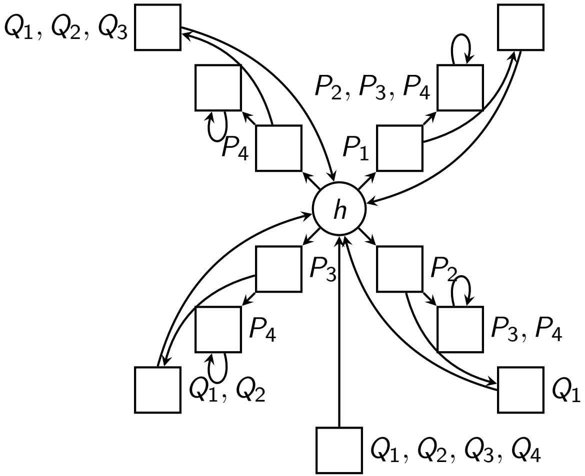
Martin Zimmermann

Saarland University

September 26th, 2014

AVACS Meeting, Saarbrücken, Germany

Let's Play



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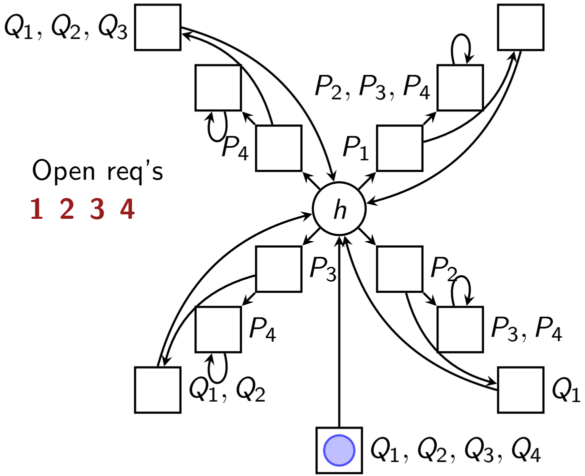

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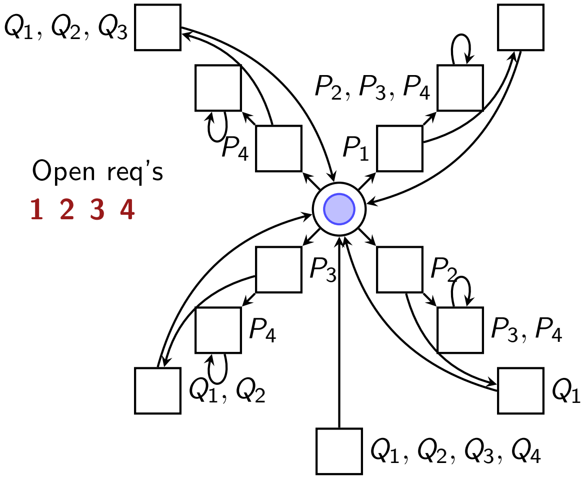

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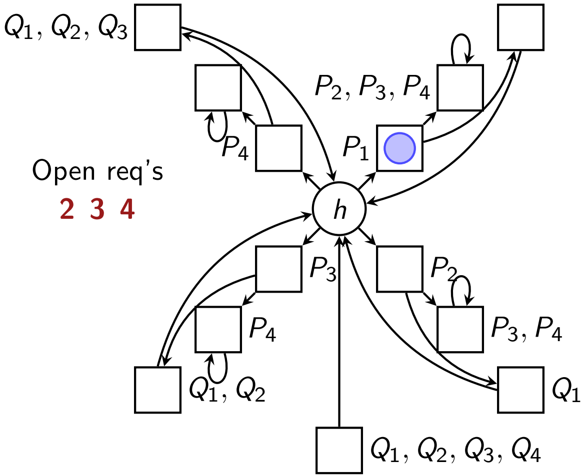

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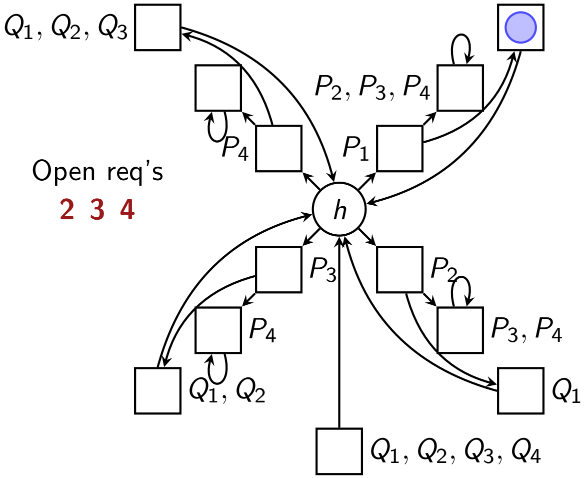

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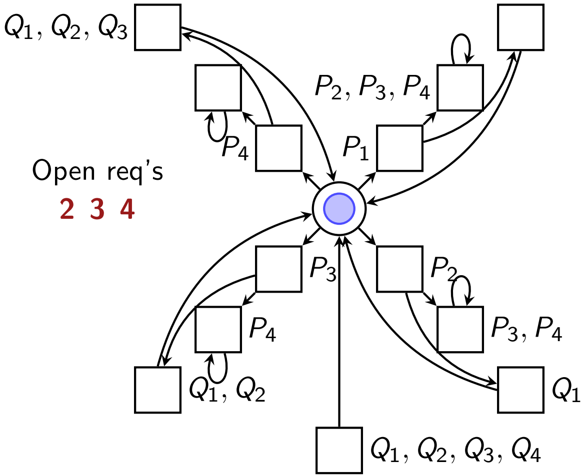

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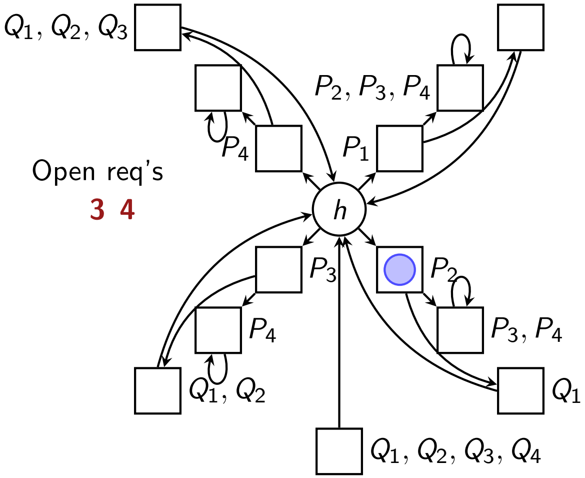

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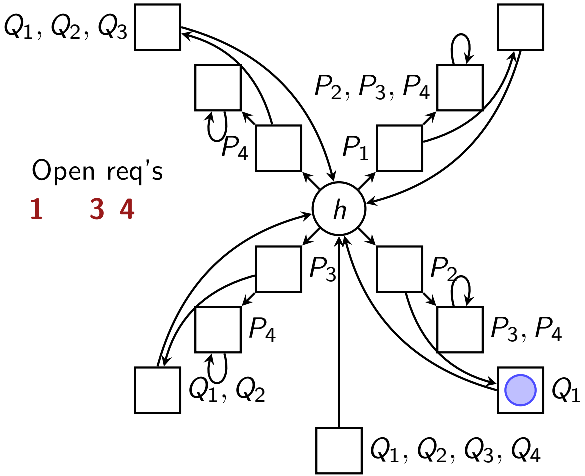

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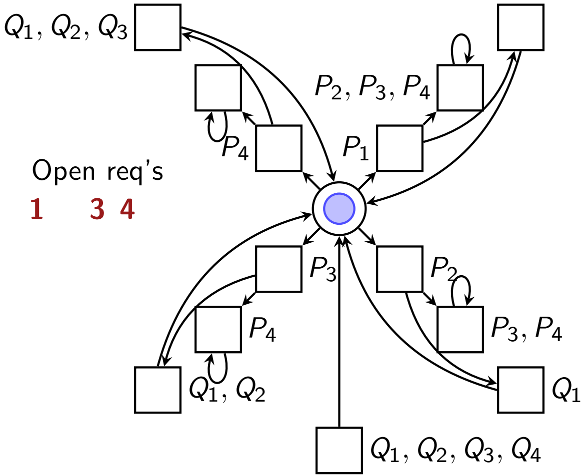

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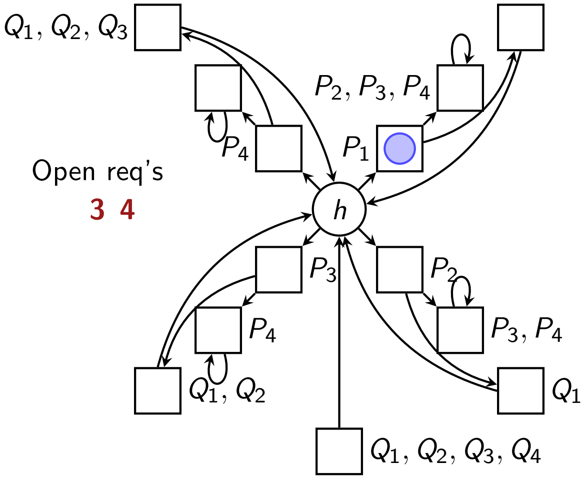

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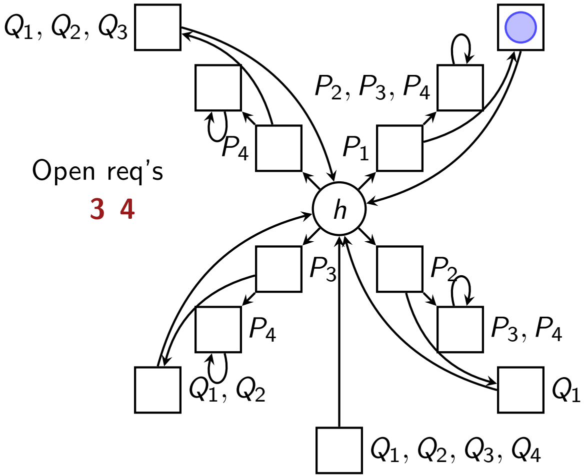

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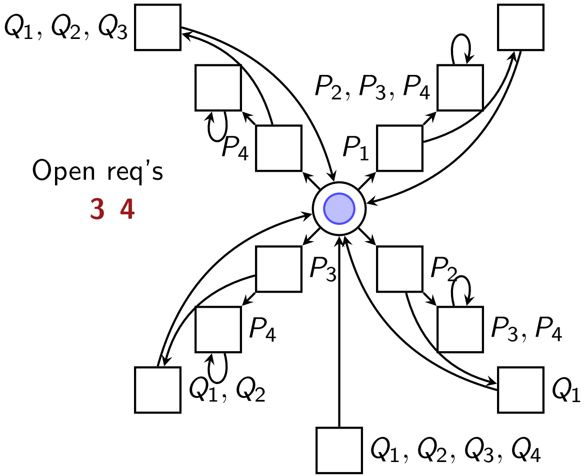

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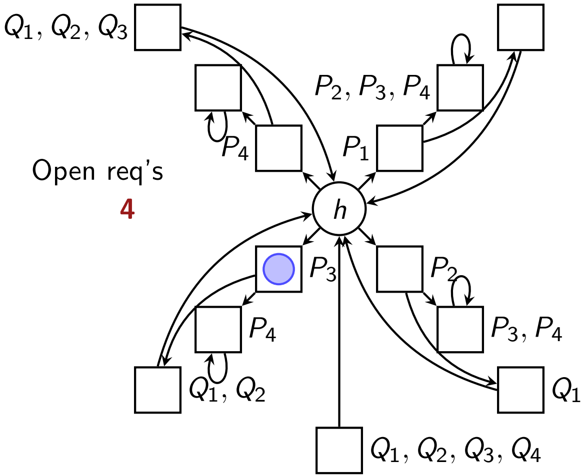

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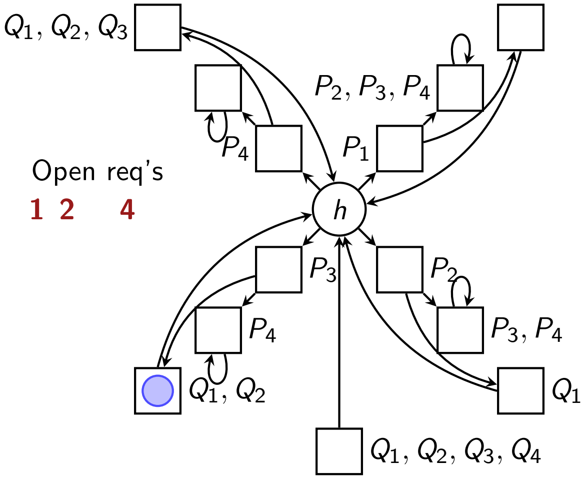

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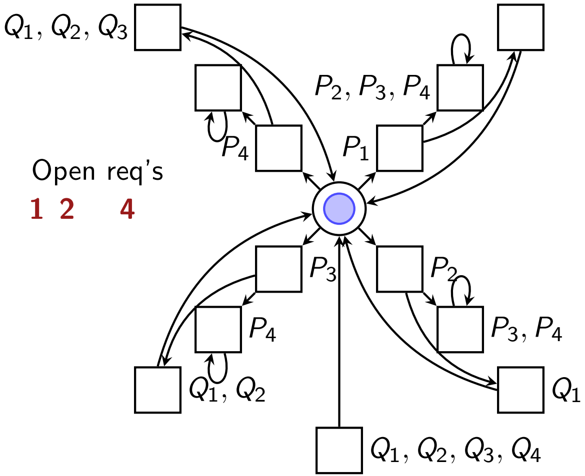

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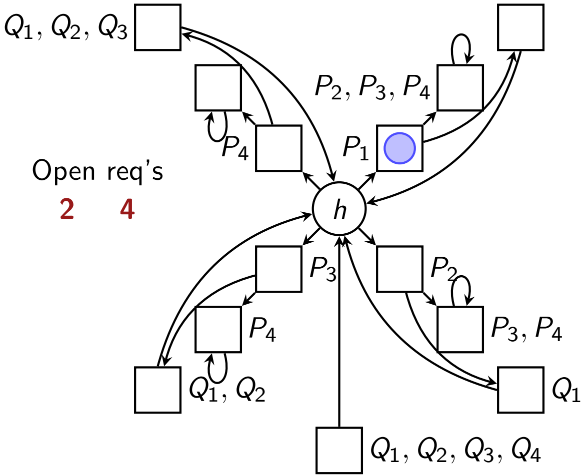

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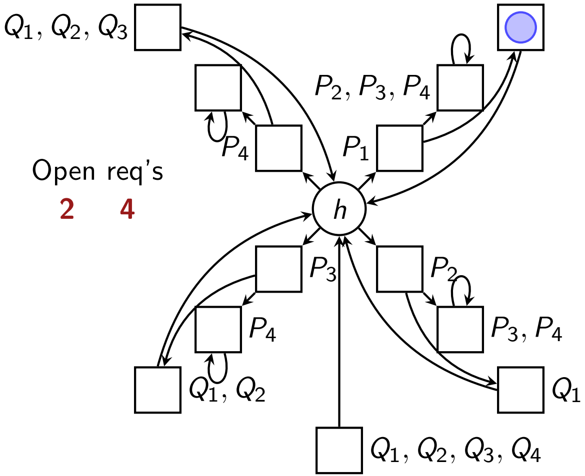

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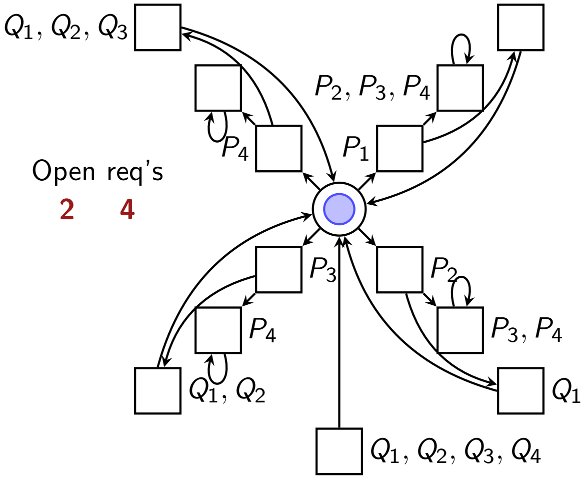

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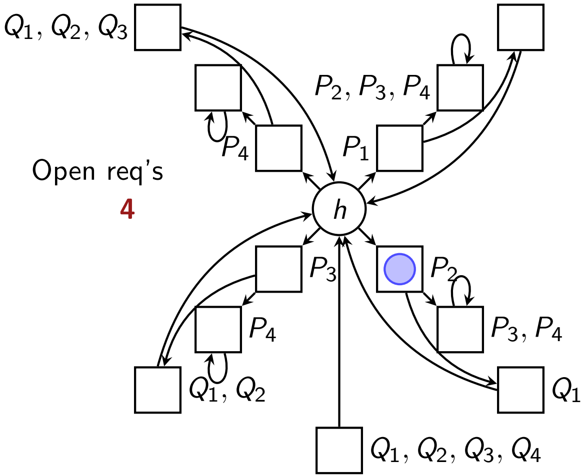

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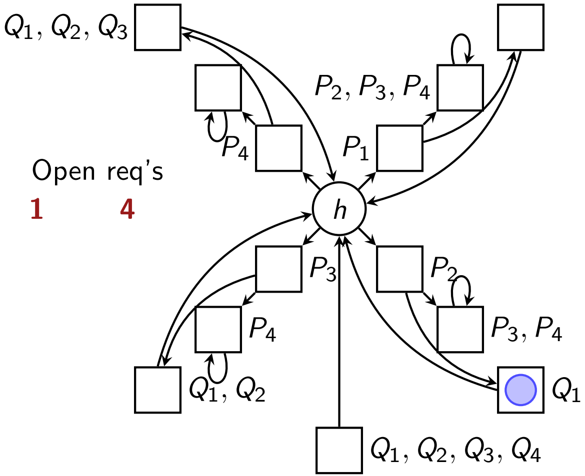

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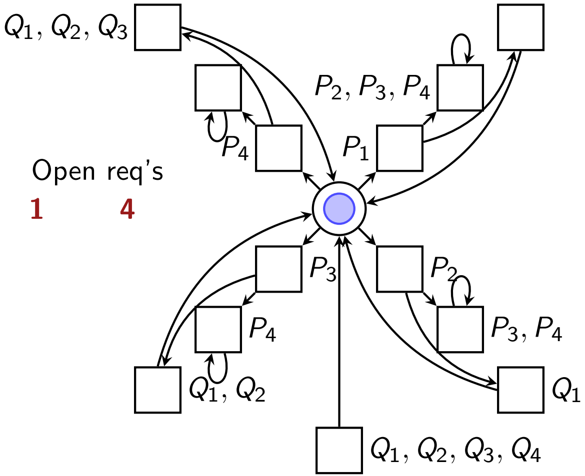

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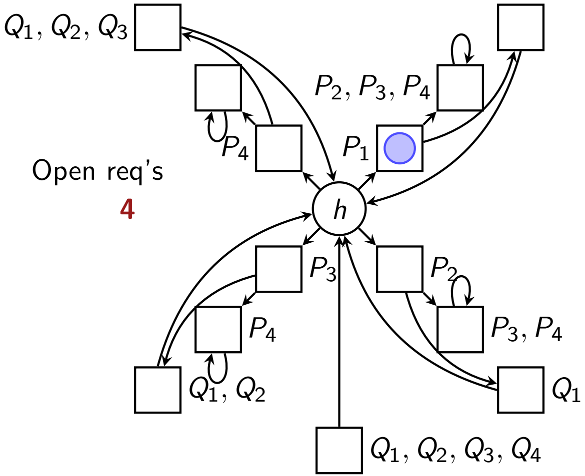

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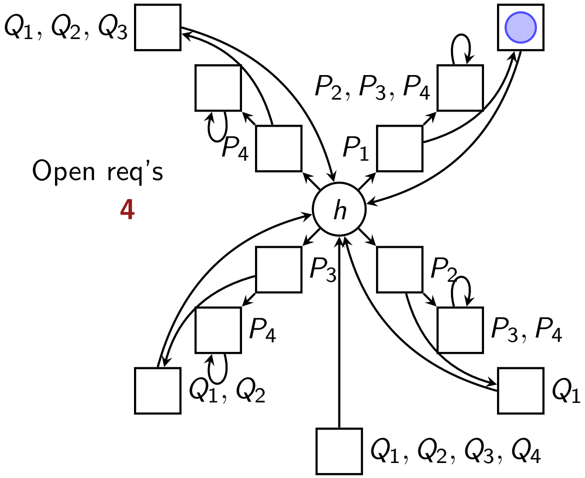

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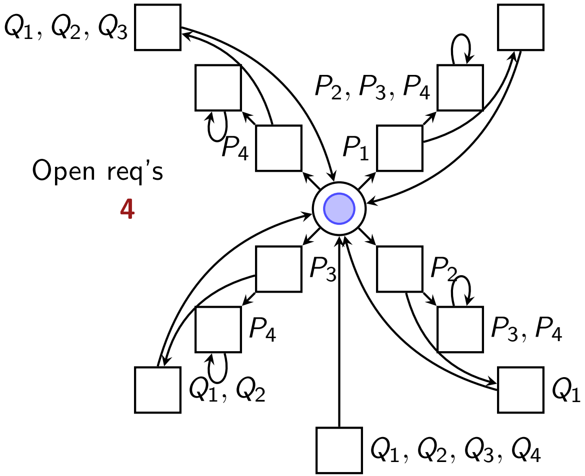

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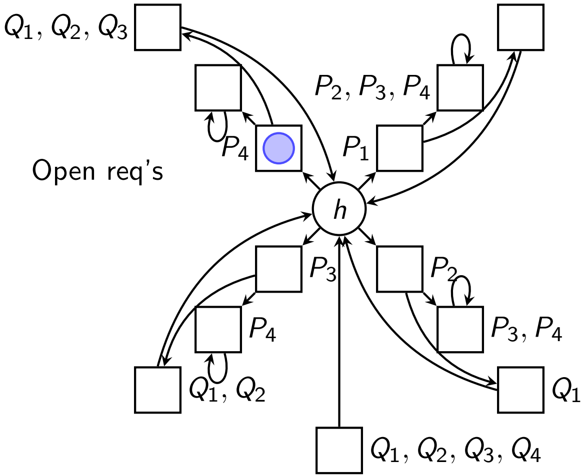

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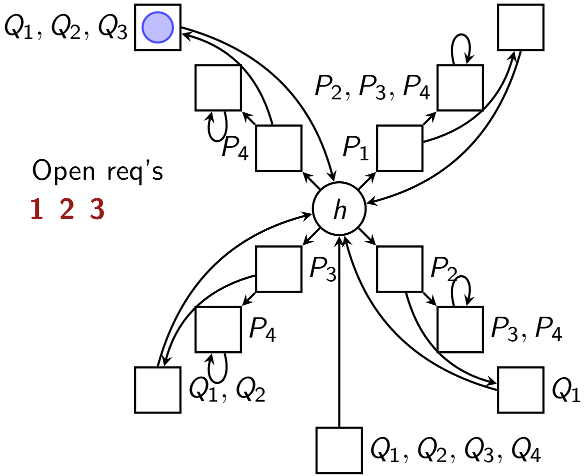

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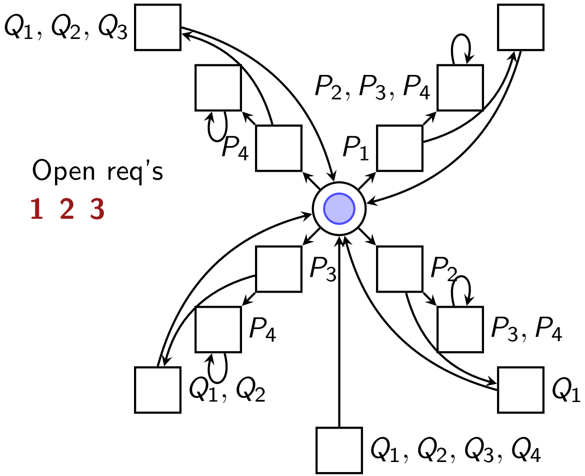

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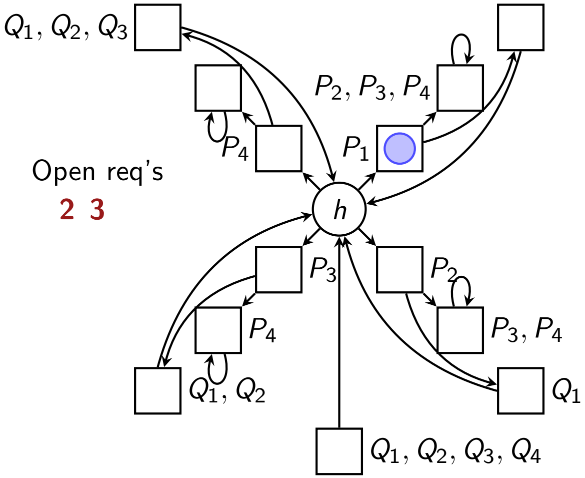

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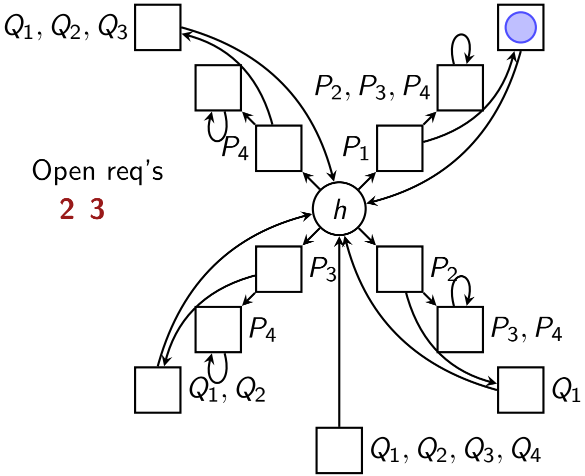

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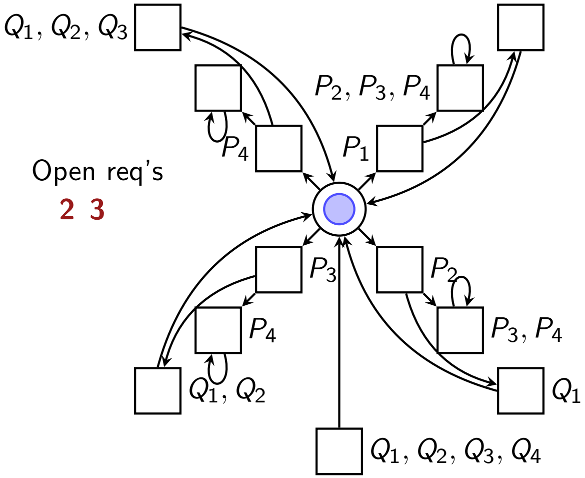

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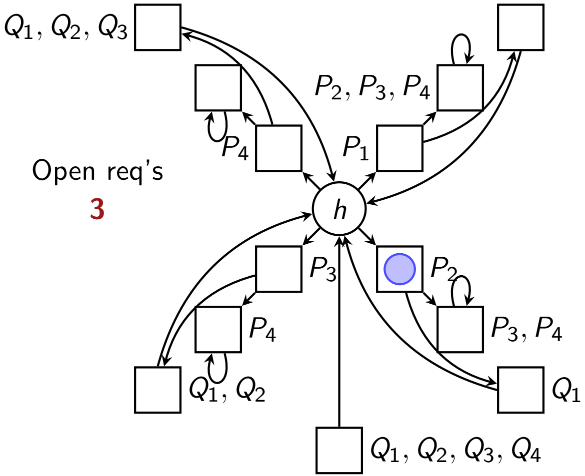

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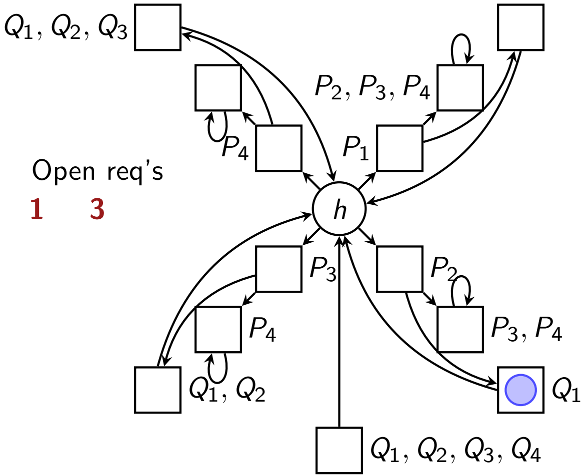

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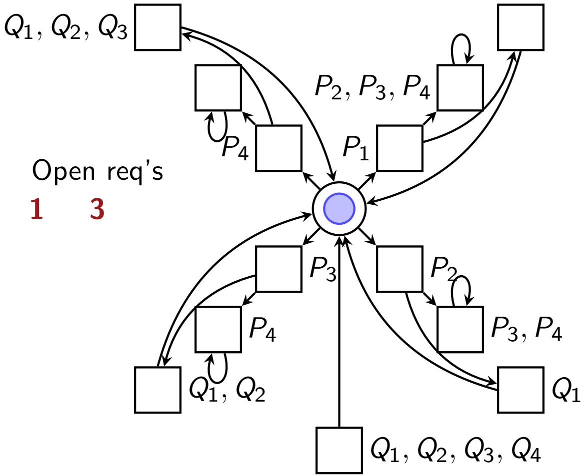

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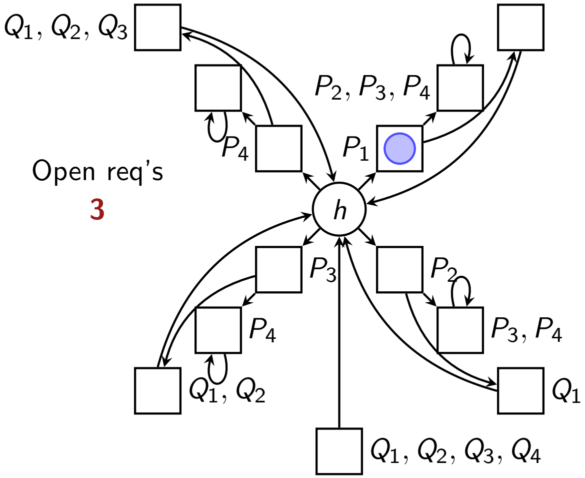

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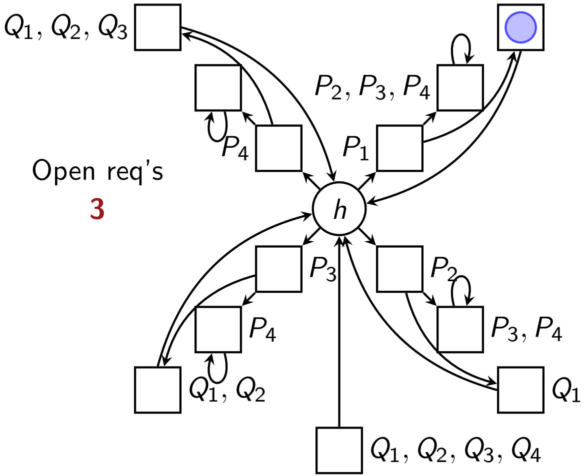

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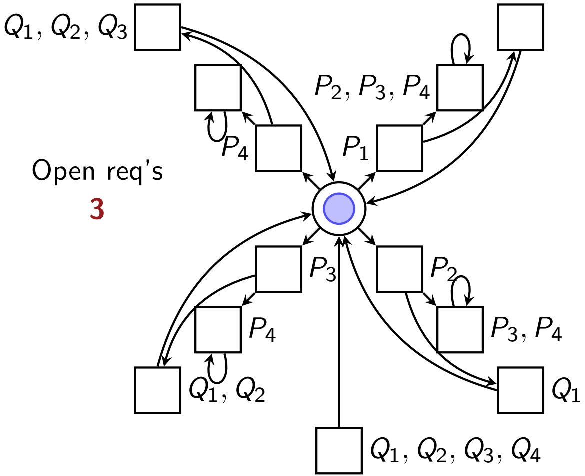

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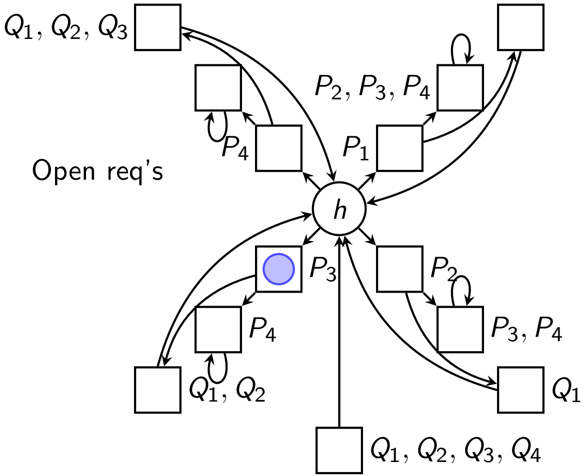

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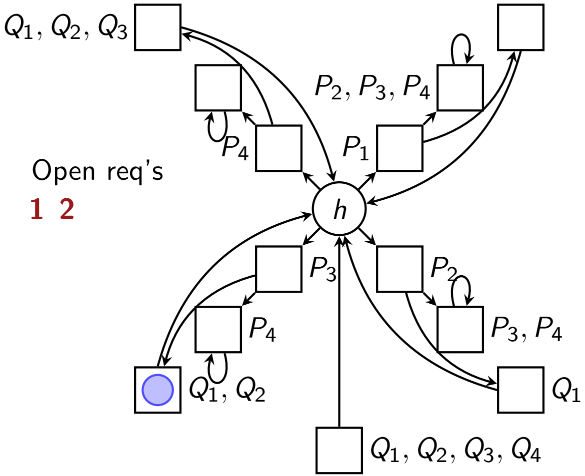

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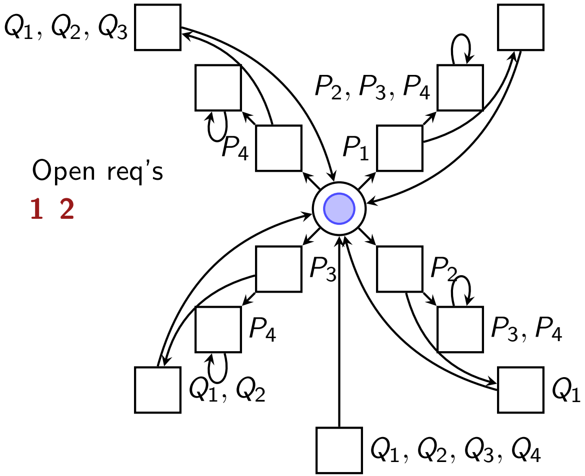

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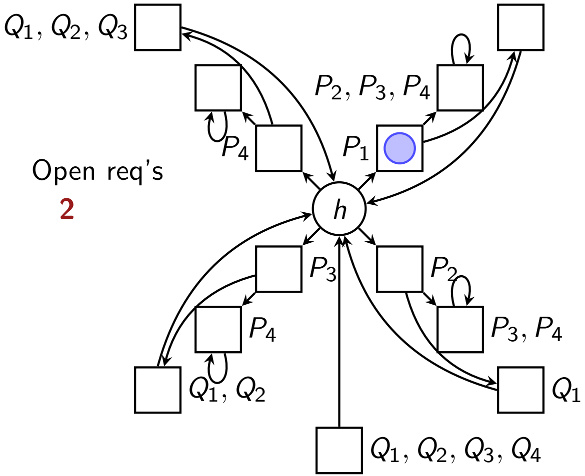

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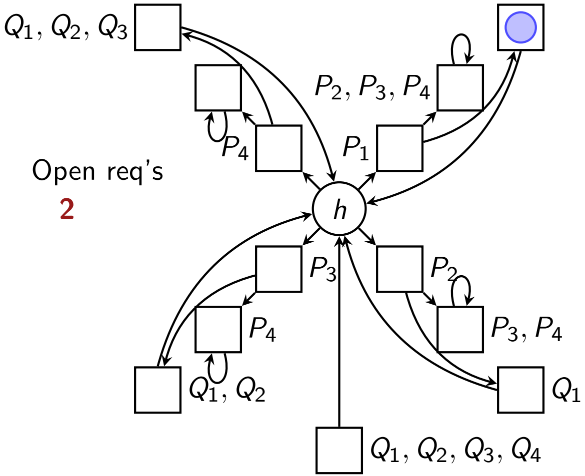

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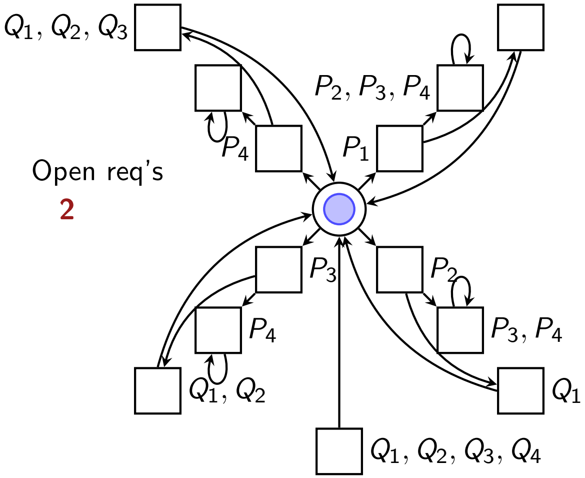

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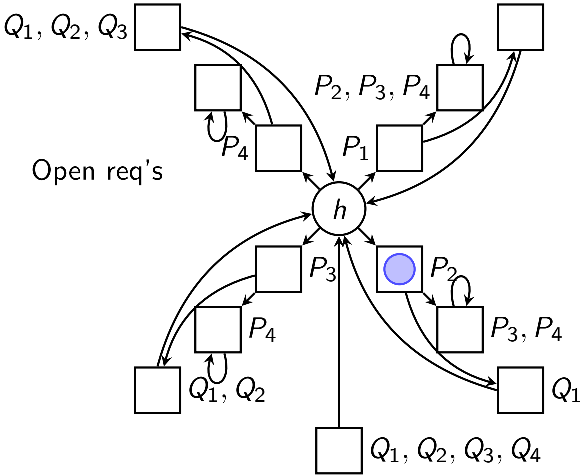

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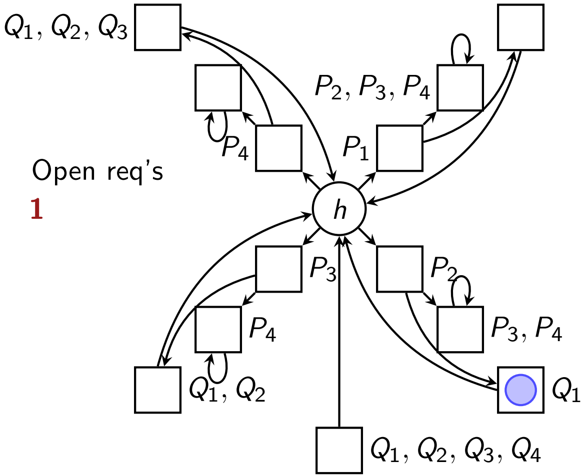

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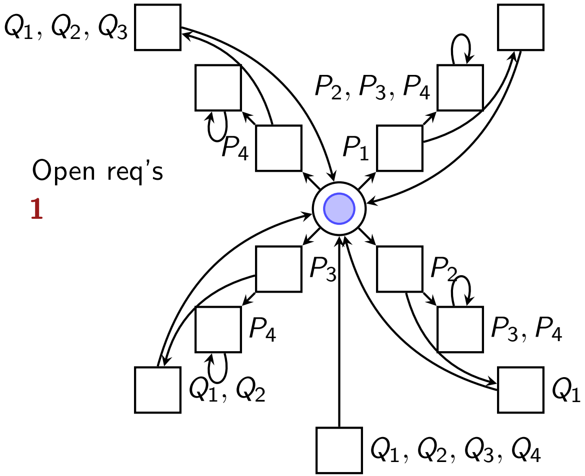

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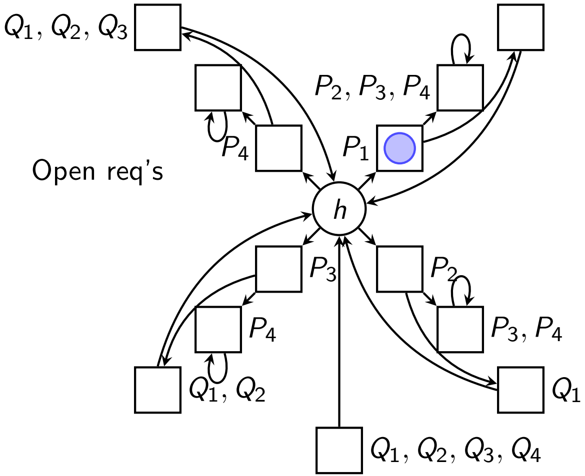

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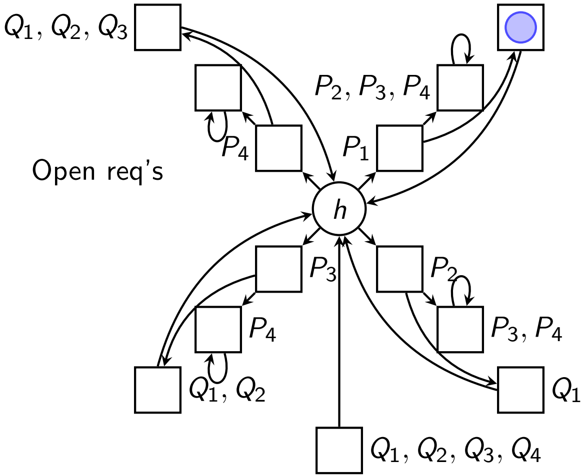

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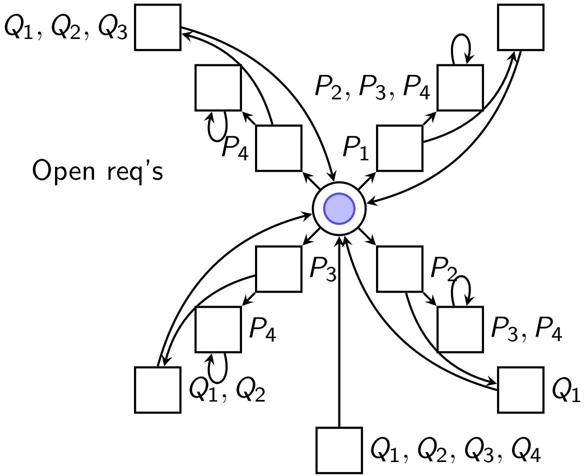

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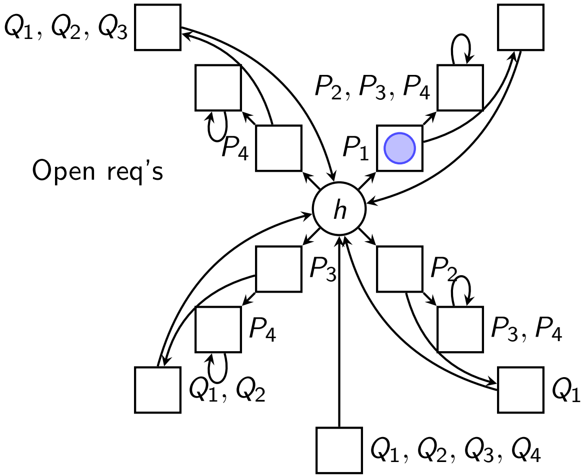

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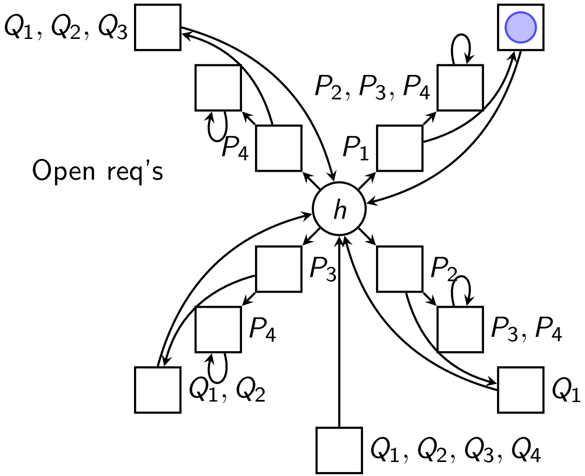

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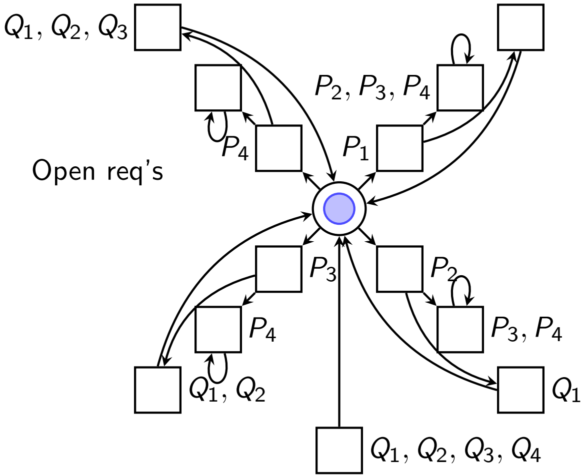

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- Winning region of Player 0:

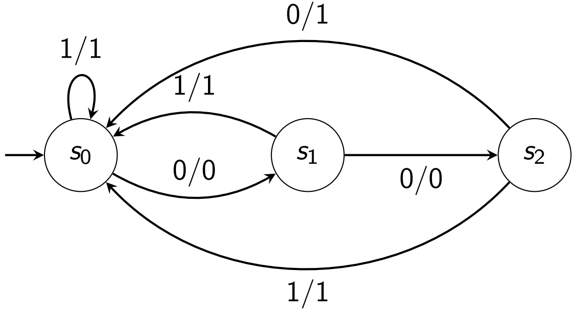
$$W_0 = \{v \mid \text{Player 0 has winning strategy from } v\}$$

Reductions and Finite-state Strategies

- Positional Strategies: move only depends on last vertex

$$\sigma(wv) = \sigma(v)$$

- Finite-state strategies: implemented by DFA with output reading play prefix $\rho_0 \cdots \rho_n$ and outputting $\sigma(\rho_0 \cdots \rho_n)$.



RR Games

Request-response game (RR game): $(\mathcal{A}, (Q_j, P_j)_{j \in [k]})$ with

- arena $\mathcal{A} = (V, V_0, V_1, E)$,
- $Q_j \subseteq V$: reQuests of condition j, and
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Corollary

- *Finite-state winning strategies of size $k2^{k+1}$ for both players.*
- *Solvable in* $\textsc{Exptime}$.

Waiting Times

- $\mathrm{wt}_j(\varepsilon) = 0$, and

$$\mathrm{wt}_j(wv) = \begin{cases} 0 & \text{if } \mathrm{wt}_j(w) = 0 \text{ and } v \notin Q_j \setminus P_j, \\ & \\ & \end{cases}$$

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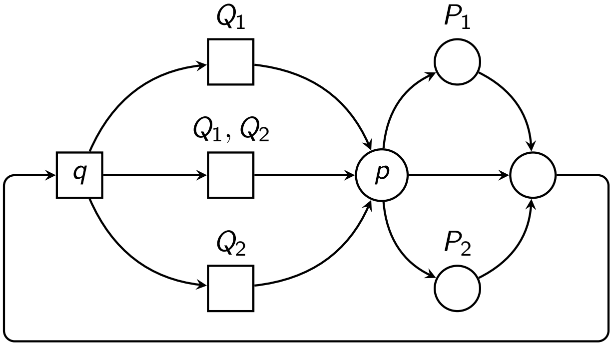
Goal:
Prove that optimal winning strategies exist and are computable.

Example



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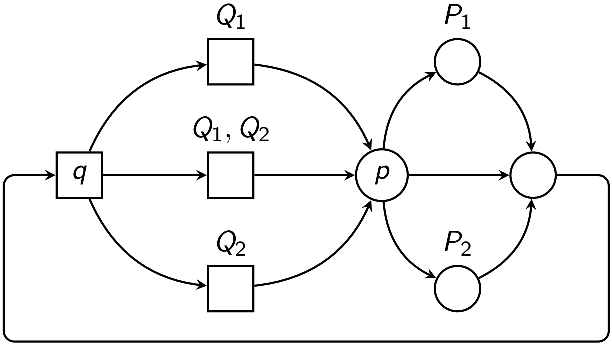

- Winning strategy σ: answer Q_1 and Q_2 alternatingly
- $\mathrm{val}(\sigma, v) = \frac{56}{10}$ for every v

Waiting Times: Upper Bounds

Lemma

Player 0 *has a winning strategy* σ *with* $\mathrm{val}(\sigma, v) \leq \sum_{j\in[k]} sk2^{k+1}$ *for every* $v \in W_0(\mathcal{G})$.

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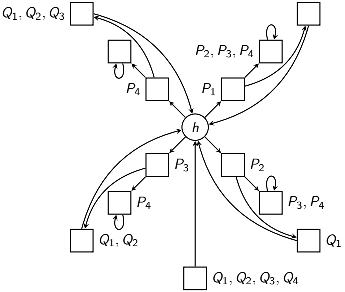

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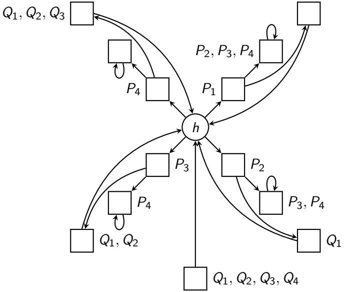

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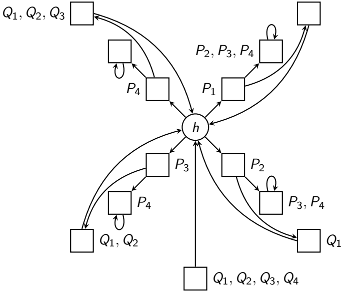

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2. Expand arena by keeping track of waiting time vectors up to bound from 1.). RR-values equal to mean-payoff condition.
   - Optimal strategy for mean-payoff yields optimal strategy for RR game.

Dickson's Lemma

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$$\big(n\big)\,\big(n-1\big)\,\big(n-2\big)\cdots\big(0\big)$$

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Lemma (Czerwiński, Gogac, Kopczyński '14)

Lower bound: $2^{2^{k/2}}$.

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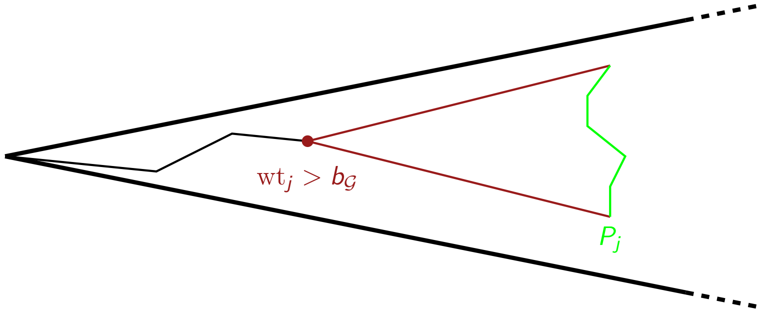

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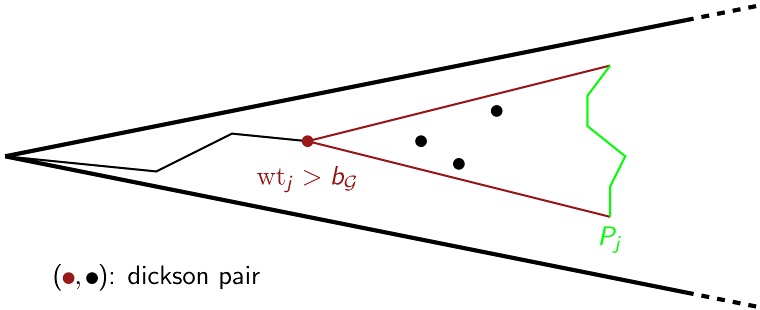

$(\bullet, \bullet)$: dickson pair

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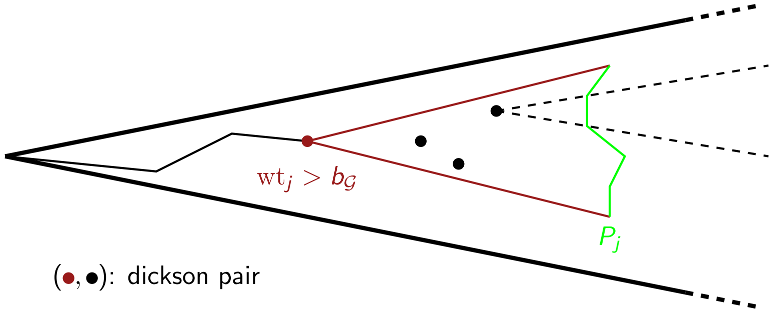

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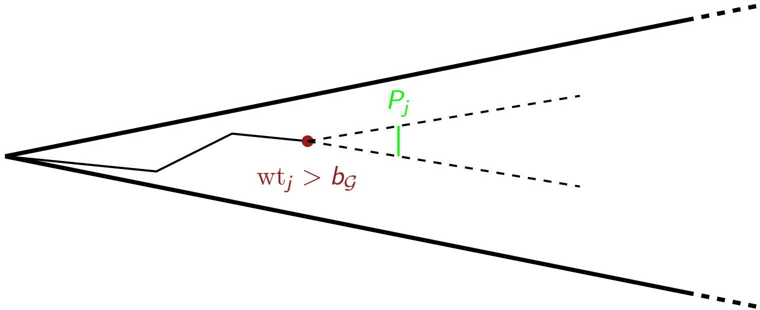

Mean-Payoff Games

Mean-payoff game: $\mathcal{G} = (\mathcal{A}, w)$ with $w \colon E \to \{-W, \ldots, W\}$.

- Given $\rho = \rho_0 \rho_1 \rho_2 \cdots$ define value for
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For every mean-payoff game there exist positional strategies σ_{opt} for Player 0 and τ_{opt} for Player 1 and values $\nu(v)$ such that

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Strategies and values are computable in pseudo-polynomial time.

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From RR Games to Mean-Payoff Games

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- Obtain mean-payoff game $\mathcal{G}' = (\mathcal{A}\times\mathfrak{A}, w)$.

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Let $\rho = \rho_0 \rho_1 \rho_2 \cdots$ be a play in $\mathcal{G}$, ρ' the corresponding one in $\mathcal{G}'$.

1. *ρ' does not reach $\perp$: $\mathrm{val}(\rho) = \nu_0(\rho') < 1 + \sum_{j \in [k]} t_{\mathsf{max}_j}$.*

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Proof of Main Theorem

RR game $\mathcal{G}$, mean-payoff game $\mathcal{G}'$.

- $\boldsymbol{\sigma}$ uniformly bounds the waiting times in $\mathcal{G}$ by t_{max_j}.
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- assume $\widehat{\boldsymbol{\sigma}}_{\mathrm{opt}}$ is strictly better.
- Turn into $\widehat{\boldsymbol{\sigma}}'_{\mathrm{opt}}$ for $\mathcal{G}'$, which is strictly better than $\boldsymbol{\sigma'_{\mathrm{opt}}}$.

Contradiction.

Conclusion

Optimal strategies for RR games exist and can be effectively computed.

- But they are larger than arbitrary strategies.
- Is this avoidable or is there a price to pay for optimality?
- What about heuristics, approximation algorithms?

Same questions can be asked for other winning conditions and other combinations of quality measures.