Degrees of Lookahead in Context-free Infinite Games*

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Motivation: We extend regular infinite games in two directions: context-free winning conditions and the possibility for one player to delay her moves. Walukiewicz showed that games with deterministic context-free winning conditions can be solved in exponential time [4] and Hosch and Landweber [3] (and later Holtmann, Kaiser, and Thomas [2]) considered delay games: in such a game, one of the players can postpone her moves for some time, thereby obtaining a lookahead on the moves of her opponent. This allows her to win some games which she loses without lookahead, e.g., if her first move depends on the third move of her opponent. On the other hand, there are simple winning conditions that cannot be won with any finite lookahead, e.g., if her first move depends on all of the infinitely many moves of her opponent.

We consider games in which the two players pick letters from alphabets Σ_I and Σ_O, respectively, thereby producing two infinite words α and β. Thus, a strategy for the second player induces a mapping $\sigma\colon \Sigma_I^\omega \to \Sigma_O^\omega$. It is winning if $(\alpha, \sigma(\alpha))$ is in the winning condition $L \subseteq \Sigma_I^\omega \times \Sigma_O^\omega$ for every α. In this case, we say that σ uniformizes L. In the classical setting, in which the players alternatingly pick letters, the n-th letter of $\sigma(\alpha)$ depends only on the first n letters of α. A strategy with bounded lookahead induces a Lipschitz-continuous function σ (in the Cantor topology on Σ^ω) and a strategy with arbitrary lookahead induces a continuous function (or equivalently, a uniformly continuous function).

Stated in these terms, Hosch and Landweber proved the decidability of the uniformization problem for regular relations by Lipschitz-continuous functions. Holtmann, Kaiser, and Thomas proved the equivalence of the existence of a continuous uniformization function and the existence of a Lipschitz-continuous uniformization function for regular relations. They observe that this equivalence does not hold for context-free winning conditions by giving an example in which every other move has to be postponed, i.e., the lookahead grows linearly. They ask whether the winner of such a game can be determined effectively and what kind of lookahead is necessary to win. We answer these questions in [1].

Definitions: Let $f\colon \mathbb{N} \to \mathbb{N}_+$ and let $L \subseteq (\Sigma_I \times \Sigma_O)^\omega$. The game $\Gamma_f(L)$ is played by two players (the input player I and the output player O) in rounds $i = 0, 1, 2, \ldots$ as follows: in round i, Player I picks a word $u_i \in \Sigma_I^{f(i)}$, then Player O picks one letter $v_i \in \Sigma_O$. The sequence $(u_0, v_0), (u_1, v_1), (u_2, v_2), \ldots$

* A full version of this abstract containing all proofs will be published as [1].

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induces two infinite words $\alpha = u_0u_1u_2\cdots$ and $\beta = v_0v_1v_2\cdots$. Player O wins the play if and only if $\binom{\alpha(0)}{\beta(0)}\binom{\alpha(1)}{\beta(1)}\binom{\alpha(2)}{\beta(2)}\cdots$ is in L.

For a delay function $f\colon \mathbb{N} \to \mathbb{N}_+$ define its *distance function* d_f by $d_f(i) = \left(\sum_{j=0}^{i} f(j)\right) - (i+1)$. We say that f is a *constant delay function* with delay d, if $d_f(i) = d$ for all i; f is a *linear delay function* with delay $k > 0$, if $d_f(i) = (i+1)(k-1)$ for all i; and we say that f is an *elementary delay function*, if $d_f \in \mathcal{O}(\exp_k)$ for some k-fold exponential function $\exp_k$.

To specify winning conditions, we consider deterministic *parity pushdown automata* (parity-DPDA) working on infinite words. It is easy to see that $\Gamma_f(L)$ is determined, if L is recognized by a parity-DPDA.

Results: By encoding the delay into the winning condition, we obtain a decidability result.

Theorem 1. *The following problem is decidable:*
Input: *Parity-DPDA $\mathcal{A}$ and $f\colon \mathbb{N} \to \mathbb{N}_+$ such that $\{i \mid f(i) \neq 1\}$ is finite.*
Question: *Does Player O win $\Gamma_f(L(\mathcal{A}))$?*

However, the condition "$\{i \mid f(i) \neq 1\}$" marks exactly the border between decidability and undecidability. A set $\mathcal{F}$ of delay functions is *bounded*, if there exists a $d \in \mathbb{N}$ such that for every $f \in \mathcal{F}$ and every $i \in \mathbb{N}$ we have $d_f(i) \leq d$.

Theorem 2. *Let $\mathcal{F}$ be a set of delay functions. The following problem is decidable if and only if $\mathcal{F}$ is bounded:*
Input: *Parity-DPDA $\mathcal{A}$.*
Question: *Does there exist an $f \in \mathcal{F}$ such that Player O wins $\Gamma_f(L(\mathcal{A}))$?*

As a corollary, we obtain that the problem of determining whether there is a delay function f such that Player O wins $\Gamma_f(L(\mathcal{A}))$ is undecidable. This result also holds, if we restrict the question to constant or to linear delay functions, as these sets of delay functions are unbounded. Furthermore, the question is undecidable even for a fixed linear delay function. Next, we consider lower bounds on the delay necessary to win a context-free delay game.

Theorem 3. *There exists a parity-DPDA $\mathcal{A}$ and a delay function f such that Player O wins $\Gamma_f(L(\mathcal{A}))$, but Player I wins $\Gamma_{f'}(L(\mathcal{A}))$ for every elementary delay function f'.*

The undecidability results and the lower bound already hold for visibly one-counter parity-DPDA with weak-parity acceptance condition.

References

1. Fridman, W., Löding, C., Zimmermann, M.: Degrees of lookahead in context-free infinite games. In: CSL 2011. To appear.
2. Holtmann, M., Kaiser, Ł., Thomas, W.: Degrees of lookahead in regular infinite games. In: Ong, L., (ed), FOSSACS 2010. LNCS, vol. 6014, pp. 252–266, Springer, Heidelberg (2010)
3. Hosch, F., Landweber, L.H.: Finite delay solutions for sequential conditions. In: Nivat, M. (ed.), ICALP 1972, pp. 45–60. North-Holland, Amsterdam (1972)
4. Walukiewicz, I.: Pushdown processes: games and model-checking. Inf. Comput., 164(2), 234–263 (2001)