Playing Pushdown Parity Games in a Hurry

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Introduction. Infinite two-player games on graphs are a powerful tool to model, verify, and synthesize open reactive systems and are closely related to fixed-point logics. The winner of a play in such a game typically emerges only after completing the whole (infinite) play. Despite this, McNaughton became interested in playing infinite games in finite time, motivated by his belief that "infinite games might have an interest for casual living room recreation" [3].

As playing infinitely long is impossible for human players, McNaughton introduced scoring functions for Muller games. Each of these functions is associated to one of the two players, so it makes sense to talk about the scores of a player. The scoring functions are updated after every move and describe the progress a player has made towards winning the play. However, as soon as a scoring function reaches its predefined threshold, the game is stopped and the player whose score reached its threshold first is declared to win this (now finite) play.

By applying finite-state determinacy of Muller games, McNaughton showed that a Muller game and a finite-duration variant with a factorial threshold score have the same winner. Thus, the winner of a Muller game can be determined by solving a finite reachability game, which is much simpler to solve, albeit doubly-exponentially larger than the original Muller game. This result was improved by showing that the finite-duration game with threshold three always has the same winner as the original Muller game [1] and by a (score-based) reduction from a Muller game to a safety game whose solution not only yields the winner of the Muller game, but also a new kind of memory structure implementing a permissive winning strategy [4], i.e., the most general non-deterministic strategy that prevents the losing player from reaching a certain score. In [2], we begin to extend these results to parity games played on configuration graphs of pushdown systems. This abstract reports on first results.

Playing Pushdown Parity Games in Finite Time. We consider (min-) parity games of the form (G, col), where $G = (V, V_0, V_1, E, v_{\mathrm{in}})$ is a configuration graph of a pushdown system $\mathscr{P} = (Q, \Gamma, \Delta, q_{\mathrm{in}})$. Here, the color of a vertex (which is a configuration) and its membership in V_0 or V_1 only depend on the state of the configuration and the edge relation is given by the transition relation of the system.

For a path through G, a configuration is said to be a stair configuration if no subsequent configuration of strictly smaller stack height exists in this path. Note that the last configuration of a finite path is always a stair. Let $\mathsf{reset}(v) = \varepsilon$ and $\mathsf{lastBump}(v) = v$ for $v \in V$ and for $w = w(0) \cdots w(r) \in V^+$, let $\mathsf{reset}(w) = w(0) \cdots w(l)$ and $\mathsf{lastBump}(w) = w(l+1) \cdots w(r)$ where l is the greatest position such that the stack height of $w(l)$ is smaller or equal to the stack height of $w(r)$ and $l \neq r$, i.e., l is the second largest stair of w. We use the decomposition of w into blocks induced by its stairs to define a scoring function for pushdown games. To this end, let $\mathsf{MinCol}(w) = \min\{\mathrm{col}(w(i)) \mid 0 \leq i < |w|\}$. For every color c, define the function $\mathsf{StairSc}_c \colon V^* \to \mathbb{N}$ by $\mathsf{StairSc}_c(\varepsilon) = 0$ and for $w \in V^+$ by

$$\mathsf{StairSc}_c(w) = \begin{cases} \mathsf{StairSc}_c(\mathsf{reset}(w)) & \text{if } \mathsf{MinCol}(\mathsf{lastBump}(w)) > c, \\ \mathsf{StairSc}_c(\mathsf{reset}(w)) + 1 & \text{if } \mathsf{MinCol}(\mathsf{lastBump}(w)) = c, \\ 0 & \text{if } \mathsf{MinCol}(\mathsf{lastBump}(w)) < c. \end{cases}$$

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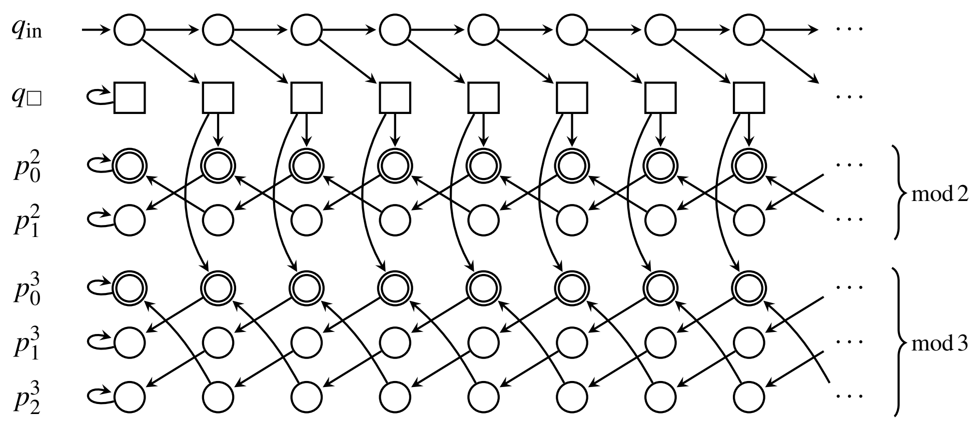
For c, the function $\mathsf{MaxStairSc}_c\colon V^* \to \mathbb{N}$ is defined by $\mathsf{MaxStairSc}_c(w) = \max_{w' \sqsubseteq w} \mathsf{StairSc}_c(w')$.

A finite-time pushdown game (G, col, k) consists of a pushdown game graph G, a coloring function col and a threshold $k \in \mathbb{N}$. A play in (G, col, k) is a finite path $w = w(0)\cdots w(r) \in V^*$ with $w(0) = v_{\mathrm{in}}$ such that $\mathsf{MaxStairSc}_c(w) = k$ for some c, and $\mathsf{MaxStairSc}_c(w(0)\cdots w(r-1)) < k$ for all c. The play w is winning for Player i if the parity of c is i. Every threshold score k is eventually reached by some stair-score function after at most 2^{k^n} moves (where n is the number of colors) and in every move, exactly one stair-score function is increased, i.e., a finite-time pushdown game is a finite zero-sum game.

Theorem 1. *Let $\mathscr{G} = (G, \mathrm{col})$ be a pushdown game and let $\mathscr{G}_k = (G, \mathrm{col}, k)$ be the corresponding finite-time pushdown game with threshold $k > |Q| \cdot |\Gamma| \cdot 2^{|Q| \cdot n} \cdot n$. Player i wins $\mathscr{G}$ if and only if he wins $\mathscr{G}_k$.*

Consider the parity game depicted below, where double-lined vertices are colored by 0, all other vertices by 1. A play in this game proceeds as follows: Player 0 picks a natural number $x > 0$ by moving the token to the configuration $(q_\Box, A^x\bot)$. If she fails to do so by staying in state q_{in} ad infinitum she loses. At $(q_\Box, A^x\bot)$ Player 1 picks a modulo counter $i \in \{2,3\}$ by moving the token to $(p_0^i, A^x\bot)$. From this configuration, a single path emanates, i.e., there is only one way to continue the play. Player 0 wins if and only if $x \bmod i = 0$. Hence, Player 0 has a winning strategy for this game by moving the token to some non-zero multiple of 6, i.e., Player 0 wins (G, col). Now, let $k \leq 6$. If Player 0 reaches $(q_{\mathrm{in}}, A^{k-1}\bot)$, she loses in the finite-time pushdown game (G, col, k), since then Player 1 obtains stair-score k. On the other hand, if she moves the token to a configuration $(q_\Box, A^j\bot)$ for some $j \leq k-1$, she loses as well, since j is not equal to 0 modulo 2 or 3. Hence, Player 0 loses (G, col, k) for every $k < 6$.

In general, using the first n prime numbers we obtain almost matching lower bounds for the minimal threshold that guarantees an equivalent finite-time pushdown game, since the product of the first n primes (which is the score Player 1 can enforce) is exponentially larger than their sum (the size of the arena).



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