Playing Muller Games in a Hurry [*]

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Abstract

This work studies the following question: can plays in a Muller game be stopped after a finite number of moves and a winner be declared? A criterion to do this is sound if Player 0 wins an infinite-duration Muller game if and only if she wins the finite-duration version. A sound criterion is presented that stops a play after at most 3^n moves, where n is the size of the arena. This improves the bound $(n! + 1)^n$ obtained by McNaughton and the bound $n! + 1$ derived from a reduction to parity games.

1 Introduction

In an infinite game, two players move a token through a finite graph thereby building an infinite path. The winner is determined by a partition of the infinite paths through the arena into the paths that are winning for Player 0 or winning for Player 1, respectively. Many winning conditions in the literature depend on the vertices that are visited infinitely often, i.e., the winner of a play cannot be determined after any finite number of steps. We are interested in the following question: is it nevertheless possible to give a criterion to define a finite-duration variant of an infinite game? Such a criterion has to stop a play after a finite number of steps and then declare a winner based on the finite play constructed thus far. It is sound if Player 0 has a winning strategy for the infinite duration game if and only if she has a winning strategy for the finite duration game.

McNaughton considered the problem of playing infinite games in finite time from a different perspective. His motivation was to make infinite games suitable for "casual living room recreation" [5]. As human players cannot play infinitely long, he envisions a referee that stops a play at a certain time and declares a winner. The justification for declaring a winner is that "if the play were to continue with each [player] playing forever as he has so far, then the player declared to be the winner would be the winner of the infinite play of the game" [5].

Besides this recreational aspect of infinite games there are several interesting theoretical questions that motivate investigating this problem. If there exists a sound criterion to stop a play after at most n steps, this yields a simple algorithm to determine the winner of the infinite game: the finite duration game can be seen as a reachability game on a finite tree of depth at most n that is won by the same player that wins the infinite duration game. There exist simple and efficient algorithms to determine the winner in reachability games on trees. Furthermore, a positive answer to the question whether a winning strategy for the reachability game can be turned into a (small finite-state) winning strategy should yield better

results in the average (although not in the worst case) than game reductions, which ignore the structure of the arena.

Consider the following criterion: the players move the token through the arena until a vertex is visited for the second time. An infinite play can then be obtained by assuming that the players continue to play the cycle that they have constructed. Then, the winner of the infinite play is declared to be the winner of the finite play. If the game is determined with positional strategies for both players, then this procedure is correct: if a player has a winning strategy for the infinite game, which can be assumed to be positional, then she can use the same strategy to win the finite version of the game and vice versa.

Therefore, McNaughton proposes that we should consider games that are in general not positionally determined. Here, the first loop of a play is typically not an indicator of how the infinite play evolves, as the memory allows a player to make different decisions when a vertex is seen again. Therefore, the players have to play longer before the play can be stopped and analyzed.

McNaughton considers Muller games, which are games of the form $(G, \mathscr{F}_0, \mathscr{F}_1)$, where G is a finite arena and $(\mathscr{F}_0, \mathscr{F}_1)$ is a partition of the set of vertices. Player i wins a play, if the set of vertices visited infinitely often by this play is in $\mathscr{F}_i$. Muller winning conditions allow us to express all other winning conditions that depend only in the infinity set of a play (e.g., Büchi, co-Büchi, parity, Rabin, and Streett conditions).

To give a sound criterion for Muller games, McNaughton defines for every set of vertices F a scoring function Sc_F that keeps track of the number of times the set F was visited entirely since the last visit of a vertex that is not in F. In an infinite play, the set of vertices seen infinitely often is the unique set F such that Sc_F will tend to infinity with being reset to 0 only finitely often.
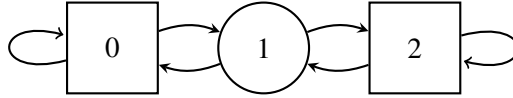

Figure 1: The arena G.

Let G be the arena in Figure 1 (Player 0's vertices are shown as circles and Player 1's vertices are shown as squares) and the Muller game $\mathscr{G} = (G, \mathscr{F}_0, \mathscr{F}_1)$ with $\mathscr{F}_0 = \{\{0,1,2\}, \{0\}, \{2\}\}$. In the play 100122121 we have that the score for the set $\{1,2\}$ is 3, as it was seen thrice (i.e., with the infixes 12, 21, and 21). Note that the order of the visits to the elements of F is irrelevant and that it is not required to close a loop in the arena. The following winning strategy for Player 0 bounds the scores of Player 1 by 2: arriving from 0 at 1 move to 2 and vice versa. However, Player 0 cannot avoid a score of 2 for her opponent, as either the play prefix 1001 or 1221 is consistent with every winning strategy.

By using finite-state determinacy of Muller games, McNaughton suggests that the criterion should stop a play after a score of $|F|! + 1$ for some set F is reached. He shows that picking the winner to be the Player i such that $F \in \mathscr{F}_i$ is indeed sound.

Applying finite-state determinacy one can also show that one can soundly declare a winner after at most $|G|! + 1$ steps, as a repetition of a memory state has occurred after that many steps. Note that for *large* sets F, it could take far more than $|G|! + 1$ steps to reach a score of $|F|! + 1$, as scores can increase slowly or can even be reset to 0. However, to decide whether a memory state repetition has occurred, it might be necessary to compute the complete memory structure for the given game, which is of size $|G|!$. Keeping track of scores is much simpler, as they can be computed on the fly while the play is being played. Also, there are at most $|G|$ sets F with non-zero score.

Our contribution. We show that declaring the winner of a play as soon as the score 3 is reached for the first time is a sound criterion. We complement this by proving that a score of 3 is reached after at most $3^{|G|}$ steps. Hence, we obtain a better bound than $|G|! + 1$, which was derived from waiting for repetitions of memory states.

Our results are obtained by using Zielonka's algorithm [7] (a reinterpretation of an earlier algorithm by McNaughton [4]) for computing winning regions in Muller games. We carefully define a winning strategy that bounds the scores of the opponent by 2. In the example above, the winning player cannot avoid a score of 2 for the opponent. Hence, in this sense our result is optimal.

Related work. Usually, the quality of a strategy is measured in terms of memory needed to implement it. However, there are other natural quality measures of winning strategies. In [1], the authors study a strengthening of parity (and Streett) objectives, which require that there is some bound between the occurrences of even colors. Another application of this concept appears in work on request response games [3], where waiting times between requests and subsequent responses are used to define the value of a play. There it is shown that time-optimal winning strategies can be computed effectively.

The maximal score achieved by the opponent in a play can be used to measure the quality of winning plays in a Muller game. Player 0 prefers plays in which the scores of her opponent are *small*.

Basic notations. An arena is denoted as $G = (V, V_0, V_1, E)$. A Muller game $(G, \mathscr{F}_0, \mathscr{F}_1)$ consists of an arena G and a partition $(\mathscr{F}_0, \mathscr{F}_1)$ of 2^V. The Zielonka tree [7] of $(\mathscr{F}_0, \mathscr{F}_1)$ is denoted by $\mathscr{Z}_{\mathscr{F}_0, \mathscr{F}_1}$. Plays are denoted by ρ. The occurrence and infinity set $\mathrm{Inf}(\rho)$ of ρ are given by $\mathrm{Occ}(\rho) = \{v \in V \mid \exists n \in \mathbb{N} \text{ such that } \rho_n = v\}$ and $\mathrm{Inf}(\rho) = \{v \in V \mid \exists^{\omega} n \in \mathbb{N} \text{ such that } \rho_n = v\}$; ρ is winning for Player i if $\mathrm{Inf}(\rho) \in \mathscr{F}_i$. Strategies are denoted as σ or τ; the set of strategies for Player i is denoted by Π_i; and $\mathrm{Play}(v, \sigma, \tau)$ denotes the unique play starting in v that is consistent with $\sigma \in \Pi_i$ and $\tau \in \Pi_{1-i}$. The winning regions are denoted as W_i.

2 The Scoring Functions

This section introduces the notions that are required to formally define finite-time Muller games.

Definition 1 (Score). For every $F \subseteq V$ we define $\mathrm{Sc}_F \colon V^+ \to \mathbb{N}$ as

$$\mathrm{Sc}_F(w) = \max\{k \in \mathbb{N} \mid \exists x_1, \ldots, x_k \in V^+ \text{ such that } \mathrm{Occ}(x_i) = F \text{ for all } i \text{ and } x_1 \cdots x_k \text{ is a suffix of } w\}.$$

For every $\mathscr{F} \subseteq 2^V$ we define

$$\mathrm{MaxSc}_{\mathscr{F}} \colon V^+ \cup V^{\omega} \to \mathbb{N} \cup \{\infty\} \text{ by } \mathrm{MaxSc}_{\mathscr{F}}(\rho) = \max_{F \in \mathscr{F}} \max_{w \sqsubseteq \rho} \mathrm{Sc}_F(w) \ .$$

Example 2. Consider the word $w = aabbaabcabcac$. The scores of the sets $\{a, b\}$ and $\{a, b, c\}$ evolve as follows:

w	a	a	b	b	a	a	b	c	a	b	c	a	c
$\mathrm{Sc}_{\{a,b\}}$	0	0	1	1	2	2	3	0	0	1	0	0	0
$\mathrm{Sc}_{\{a,b,c\}}$	0	0	0	0	0	0	0	1	1	1	2	2	2

For $\mathscr{F} = \{\{a, b\}, \{a, b, c\}\}$, we have $\mathrm{MaxSc}_{\mathscr{F}}(w) = 3$.

McNaughton proposes that scores should be used to decide the winner in a finite-time Muller game. As soon as a threshold score of k for some set F is reached, the play is stopped and Player i is declared the winner, if $F \in \mathscr{F}_i$.

Lemma 3. *Let G be an arena with vertex set V. Every $w \in V^*$ with $|w| \geq k^{|V|}$ satisfies $\mathrm{MaxSc}_{2^V}(w) \geq k$.*

Lemma 3 implies that a finite-time Muller game with threshold k must end after at most $k^{|V|}$ steps. We can also show that this bound is tight. For every $k > 0$ we give an inductive definition of a word over the alphabet $\Sigma_n = \{1, \ldots, n\}$ by $w_{(k,1)} = 1^{k-1}$ and $w_{(k,n)} = (w_{(k,n-1)}n)^{k-1}w_{(k,n-1)}$. Clearly, the word $w_{(k,n)}$ has length $k^n - 1$, and it can also be shown that $\mathrm{MaxSc}_{2^{\Sigma_n}}(w) < k$.

Finally, to declare a unique winner in every finite-time Muller game we must exclude the case where there are two sets such that both sets hit score k at the same time.

Lemma 4 ([5]). *Let $k, l \geq 2$, let $F, F' \subseteq V$, let $w \in V^*$ and $v \in V$ such that $\mathrm{Sc}_F(w) < k$ and $\mathrm{Sc}_{F'}(w) < l$. If $\mathrm{Sc}_F(wv) = k$ and $\mathrm{Sc}_{F'}(wv) = l$, then $F = F'$.*

3 Finite-time Muller Games

We are now in a position to define a finite-time Muller game. Such a game $\mathscr{G} = (G, \mathscr{F}_0, \mathscr{F}_1, k)$ consists of an arena $G = (V, V_0, V_1, E)$, a partition $(\mathscr{F}_0, \mathscr{F}_1)$ of 2^V, and a threshold $k \geq 2$. By Lemma 3 we have that every infinite play must reach score k for some set F after a bounded number of steps. Therefore, we define a play for the finite-time Muller game to be a finite path $w = w_0 \cdots w_n$ with $\mathrm{MaxSc}_{2^V}(w_0 \cdots w_n) = k$, but $\mathrm{MaxSc}_{2^V}(w_0 \cdots w_{n-1}) < k$. Due to Lemma 4, there is a unique $F \subseteq V$ such that $\mathrm{Sc}_F(w) = k$. Player 0 wins the play w if $F \in \mathscr{F}_0$ and Player 1 wins otherwise. The definitions of strategies, plays, and winning sets can be redefined for the finite games.

Zermelo [6] has shown that a game in which every play is finite is determined. Therefore, it immediately follows that finite Muller games are determined.

Lemma 5. *Finite-time Muller games are determined.*

In fact, McNaughton considered a slightly different definition of a finite-time Muller game. Rather than stopping the play when the score of a set reaches the global threshold k, his version stops the play when the score of a set F reaches $|F|! + 1$.

Theorem 6 ([5]). *If W_i is the winning region of Player i in a Muller game $(G, \mathscr{F}_0, \mathscr{F}_1)$, and W_i' is the winning region of Player i in McNaughton's finite-time Muller game, then $W_i = W_i'$.*

Now, we state our main result: the finite-time Muller game with threshold 3 is equivalent to a Muller game.

Theorem 7. *If W_i is the winning region of Player i in a Muller game $(G, \mathscr{F}_0, \mathscr{F}_1)$, and W_i' is the winning region of Player i in the finite-time Muller game $(G, \mathscr{F}_0, \mathscr{F}_1, 3)$, then $W_i = W_i'$.*

To prove Theorem 7 we use the following approach. If $\mathrm{MaxSc}_{\mathscr{F}_{1-i}}(\rho) \leq 2$ for an infinite play ρ, then there exists a prefix w of ρ that is winning for Player i in the finite-time Muller game with threshold 3. Hence, if a winning strategy for Player i in the Muller game bounds the scores of her opponent by 2, then this strategy is also winning for the finite-time Muller game with threshold 3. We will show that such a winning strategy exists. Theorem 7 then follows by determinacy of Muller games.

Lemma 8. *Player i has a winning strategy σ for her winning region W_i in a Muller game $\mathscr{G} = (G, \mathscr{F}_0, \mathscr{F}_1)$ such that $\mathrm{MaxSc}_{\mathscr{F}_{1-i}}(\mathrm{Play}(v, \sigma, \tau)) \leq 2$ for every vertex $v \in W_i$ and every $\tau \in \Pi_{1-i}$.*

The proof of Lemma 8 is achieved by defining σ based on the internal structure of the winning regions in a Muller game as computed by Zielonka's algorithm [7], which proceeds by induction over the height of the Zielonka tree $\mathscr{Z}_{\mathscr{F}_0, \mathscr{F}_1}$ of the winning condition. The strategy computed by the algorithm works by alternating between a set of recursively computed strategies that are winning for sub-games of

the full game. In Zielonka's formulation these strategies are played in a fixed cyclic order: a strategy for a certain sub-game will not be played again until the strategies for every other sub-game have been played. This strict order can be used to show that Zielonka's strategies do not bound the scores of the opponent by any constant. However, our strategy picks the sub-game strategy based on the behavior of the opponent. If the opponent has played in such a way that the score of one of his sets could rise to 2, then our strategy will pick a sub-game strategy which guarantees that the score of this set cannot rise above 2. To achieve this, one has to make sure that information about the scores of the opponent is passed while switching from one sub-game strategy to another. The details of the proof can be found in the full version [2].

4 Conclusion

We have presented a criterion to stop plays in a Muller game after a finite amount of time that preserves winning regions. Our bound $3^{|G|}$ on the length of a play improves the bound $|G|! + 1$ obtained by a reduction to parity games. Furthermore, our techniques show that the winning player can bound the scores of the opponent by 2 and that this bound 2 is tight.

However, it remains open whether a play can also be stopped after a score of 2 is reached. As the winning player cannot always avoid a score of 2 for the opponent, one has to show that the winning player always reaches a score of 2 for one of her sets before the opponent reaches score 2 for one of his sets. Another interesting direction for research is to find a construction which turns a winning strategy for a finite-time Muller game with threshold 3 (or 2, if it is equivalent) into a finite-state strategy for the original Muller game. It is conceivable that such a construction would yield memory structures that are optimized for a given arena, something which does not hold for the LAR respectively Zielonka tree structures.

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