

# Team Semantics for the Specification and Verification of Hyperproperties\*

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We develop team semantics for Linear Temporal Logic (LTL) to express hyperproperties, which have recently been identified as a key concept in the verification of information flow properties. Conceptually, we consider an asynchronous and a synchronous variant of team semantics. We study basic properties of this new logic and classify the computational complexity of its satisfiability, path, and model checking problem. Eventually, we compare its expressivity to the one of HyperLTL, another recently introduced logic for hyperproperties.

Our results show that LTL under team semantics is a viable alternative to HyperLTL, which complements the expressivity of HyperLTL and has better algorithmic properties.

## 1 Introduction

Guaranteeing security and privacy of user information is a key requirement in software development. However, it is also one of the hardest goals to accomplish. One reason for this difficulty is that such requirements typically amount to reasoning about the flow of information and relating different execution traces of the system. In particular, these requirements are no longer trace properties, i.e., properties whose satisfaction can be verified by considering each trace in isolation. For example, the property “the system terminates eventually” is satisfied if every trace eventually reaches a final state. Formally, a trace property  $\varphi$  is a set of traces and a system satisfies  $\varphi$  if each of its traces is in  $\varphi$ .

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In contrast, the property “the system terminates within a bounded amount of time” is no longer a trace property. To illustrate this, consider a system that has a trace  $t_n$  for every  $n$ , so that  $t_n$  only reaches a final state after  $n$  steps. This system does not satisfy the bounded termination property, but each individual trace  $t_n$  could also stem from a system that does satisfy it. As a result, the satisfaction of the property cannot be verified by considering each trace in isolation.

Properties with this characteristic were termed *hyperproperties* by Clarkson and Schneider [6]. Formally, a hyperproperty  $\varphi$  is a set of sets of traces and a system satisfies  $\varphi$  if its set of traces is contained in  $\varphi$ . The conceptual difference to trace properties allows specifying a much richer landscape of properties that includes information flow and trace properties (as a special case). Furthermore, one can also express specifications for symmetric access to critical resources in distributed protocols and Hamming distances between code words in coding theory [31]. However, the increase in expressiveness requires novel approaches to specification and verification.

**HyperLTL** Trace properties are typically specified in temporal logics, most prominently in Linear Temporal Logic (LTL) [30]. Verification of LTL specifications is routinely employed in industrial settings and marks one of the most successful applications of formal methods to real-life problems. Recently, this work has been extended to hyperproperties: HyperLTL, LTL equipped with trace quantifiers, has been introduced to specify hyperproperties [5]. Accordingly, a model of an HyperLTL formula is a set of traces and the quantifiers range over these traces. This logic is able to express the majority of the information flow properties found in the literature (we refer to Section 3 of [5] for a full list). The satisfiability problem for HyperLTL is undecidable [11] while the model checking problem is decidable, albeit of non-elementary complexity [5, 14]. In view of this, the full logic is too strong, but most information flow properties found in the literature can be expressed with at most one quantifier alternation and consequently belong to decidable (and tractable) fragments. Further works have studied runtime verification [2, 12], connections to first-order logic [15], provided tool support [14, 11], and presented applications to “software doping” [7] and the verification of web-based workflows [13]. In contrast, there are natural properties, e.g., bounded termination, which are not expressible in HyperLTL (which is an easy consequence of a much stronger non-expressibility result [3]).

**Team Semantics** Intriguingly, there exists another modern family of logics, *Dependence Logics* [34, 10], which operate as well on sets of objects instead of objects alone. Informally, these logics extend first-order logic (FO) by dependence atoms expressing that “the value of variable  $x$  functionally determines the value of variable  $y$ ”. Obviously, such statements only make sense when being evaluated over a set of assignments. In the language of dependence logic, sets of assignments are called *teams* and the semantics are termed *team semantics*.

In 1997, Hodges introduced compositional semantics for Hintikka’s Information-friendly logic [21]. This can be seen as the cornerstone of the mathematical framework of dependence logics. Intuitively, this semantics allows for interpreting a team as a database table. In this approach, variables of the table correspond to attributes and assignments to rows or records. In 2007, Väänänen [34] introduced his modern approach to such logics and developed team semantics as a core notion, as dependence atoms are meaningless under Tarskian semantics.

Important to note, not only functional dependence as an atomic statement has been introduced to this family of logics: *independence* [18] and *inclusion* [16] are the most prominent concepts that have been implemented in these logics. The current interest in these logics is rapidly growing and its research community aims to connect their area to a plethora of other research disciplines, e.g., linguistics [17], biology [17], game [4] and social choice theory [32], philosophy [32], and computer science [17]. In this paper, we are the first to exhibit connections to the area of

formal languages. Furthermore, team semantics found their way into modal logic [35], temporal logic [23], as well as statistics [9].

Recently, Krebs et al. [23] proposed team semantics for Computation Tree Logic (CTL), where a team consists of worlds of the transition system under consideration. They considered synchronous and asynchronous team semantics, which differ in how time evolves in the semantics of the temporal operators. They proved that satisfiability is EXPTIME-complete under both semantics while model checking is PSPACE-complete under synchronous semantics and P-complete under asynchronous semantics.

**Our Contribution** The conceptual similarities between HyperLTL and team semantics raise the question how an LTL variant under team semantics relates to HyperLTL. For this reason, we develop team semantics for LTL, analyse the computational complexity of its satisfiability and model checking problems, and subsequently compare the novel logic to HyperLTL.

When defining the logic, we follow the approach of Krebs et al. to defining team semantics for CTL: we introduce synchronous and asynchronous team semantics for LTL, where teams are now sets of traces. In particular, as a result, we have to consider potentially uncountable teams, while all previous work on model checking problems for logics under team semantics has been restricted to the realm of finite teams.

Our results are as follows: we prove that the satisfiability problem is PSPACE-complete under both semantics, by showing that the problems are both equivalent LTL satisfiability under classical semantics. Then, we consider two variants of the model checking problem. As there are uncountably many traces, we have to represent teams, i.e., sets of traces, in a finitary manner. The path checking problem asks to check whether a finite team of ultimately periodic traces satisfies a given formula. We show this problem to be in P for asynchronous semantics and PSPACE-complete for synchronous semantics. In contrast to this, in the (general) model checking problem, a team is represented by a finite transition system. Formally, given a transition system and a formula, the model checking problem asks to determine whether the set of traces of the system satisfies the formula. For asynchronous semantics, we prove the model checking problem to be PSPACE-complete, while we solve the synchronous case for the disjunction-free fragment.<sup>1</sup> Finally, we show that LTL under team semantics is able to specify properties which cannot be expressed in HyperLTL and *vice versa*.

Recall that satisfiability for HyperLTL is undecidable and model checking is of non-elementary complexity. Our results show that similar problems for LTL under team semantics have a much simpler complexity. This proposes LTL under team semantics to be a significant alternative for the specification and verification of hyperproperties that complements HyperLTL.

## 2 Preliminaries

The non-negative integers are denoted by  $\mathbb{N}$  and the power set of a set  $S$  is denoted as  $2^S$ . Throughout the paper, we fix a finite set AP of atomic propositions.

**Computational Complexity** We will make use of standard notions in complexity theory. In particular, we will use the complexity classes P and PSPACE. Most reductions used in the paper are  $\leq_m^P$ -reductions, that is, polynomial time, many-to-one reductions. However, sometimes we require a very simple kind of reduction, that is, constant-depth or  $AC^0$ -reductions. A language

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<sup>1</sup>Disjunction plays a special role in team semantics, as it splits a team into two. This raises interesting and challenging questions about the representation of such split teams by transition systems or automata. We discuss these challenges in Section 4.3.

$A$  is *constant-depth reducible* to a language  $B$ , in symbols  $A \leq_{\text{cd}} B$ , if there exists a logtime-uniform  $\text{AC}^0$ -circuit family with oracle gates for  $B$  that decides membership in  $A$ . In this context, *logtime-uniform* means that there exists a deterministic Turing machine that can check the structure of the circuit family  $\mathcal{C}$  in time  $O(\log |\mathcal{C}|)$ . For a deeper introduction into circuit theory we refer the reader to the textbook of Vollmer [36].

**Traces** A *trace* over  $\text{AP}$  is an infinite sequence from  $(2^{\text{AP}})^\omega$ ; a finite trace is a finite sequence from  $(2^{\text{AP}})^*$ . The length of a finite trace  $t$  is denoted by  $|t|$ . The empty trace is denoted by  $\varepsilon$  and the concatenation of two finite traces  $t_0$  and  $t_1$  by  $t_0t_1$ . Unless stated otherwise, a trace is always assumed to be infinite. A *team* is a (potentially infinite) set of traces.

Given a trace  $t = t(0)t(1)t(2)\cdots$  and  $i \geq 0$ , we define  $t[i, \infty) = t(i)t(i+1)t(i+2)\cdots$ , which we lift to teams  $T \subseteq (2^{\text{AP}})^\omega$  by defining  $T[i, \infty) = \{t[i, \infty) \mid t \in T\}$ . A trace  $t$  is *ultimately periodic*, if it is of the form  $t = t_0 \cdot t_1^\omega = t_0t_1t_1t_1\cdots$  for two finite traces  $t_0$  and  $t_1$  with  $|t_1| > 0$ . As a result,  $t$  is finitely represented by the pair  $(t_0, t_1)$  and we define  $\llbracket (t_0, t_1) \rrbracket = t_0t_1^\omega$ . Given a set  $\mathcal{T}$  of such pairs, we define  $\llbracket \mathcal{T} \rrbracket = \{\llbracket (t_0, t_1) \rrbracket \mid (t_0, t_1) \in \mathcal{T}\}$ , which is a team of ultimately periodic traces. We call  $\mathcal{T}$  a team encoding.

**Linear Temporal Logic** The formulas of Linear Temporal Logic (LTL) [30] are defined via the grammar

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \text{X}\varphi \mid \text{F}\varphi \mid \text{G}\varphi \mid \varphi \text{U}\varphi \mid \varphi \text{R}\varphi,$$

where  $p$  ranges over the atomic propositions in  $\text{AP}$ . The length of a formula is defined to be the number of Boolean and temporal connectives appearing in it.<sup>2</sup> Note that we only consider formulas in negation normal form, which is the reason we use the full set of temporal operators.

Next, we recall the classical semantics of LTL before we introduce team semantics. For traces  $t \in (2^{\text{AP}})^\omega$  one defines

$$\begin{aligned} t \models^c p & \quad \text{if } p \in t(0), \\ t \models^c \neg p & \quad \text{if } p \notin t(0), \\ t \models^c \psi \wedge \varphi & \quad \text{if } t \models^c \psi \text{ and } t \models^c \varphi, \\ t \models^c \psi \vee \varphi & \quad \text{if } t \models^c \psi \text{ or } t \models^c \varphi, \\ t \models^c \text{X}\varphi & \quad \text{if } t[1, \infty) \models^c \varphi, \\ t \models^c \text{F}\varphi & \quad \text{if there is a } k \geq 0 \text{ such that } t[k, \infty) \models^c \varphi, \\ t \models^c \text{G}\varphi & \quad \text{if for all } k \geq 0: t[k, \infty) \models^c \varphi, \\ t \models^c \psi \text{U}\varphi & \quad \text{if there is a } k \geq 0 \text{ such that } t[k, \infty) \models^c \varphi \text{ and for all } k' < k: t[k', \infty) \models^c \psi, \text{ and} \\ t \models^c \psi \text{R}\varphi & \quad \text{if for all } k \geq 0: t[k, \infty) \models^c \varphi \text{ or there is a } k' < k \text{ such that } t[k', \infty) \models^c \psi. \end{aligned}$$

**Team Semantics for LTL** Next, we introduce two variants of team semantics for LTL, which differ in their interpretation of the temporal operators on different traces of a team: a synchronous one (denoted by  $\models^s$ ), where time proceeds in lockstep along all traces of the team, and an asynchronous one (denoted by  $\models^a$ ). For the sake of brevity, we write  $\models^*$  whenever a definition coincides for both semantics. So, for teams  $T \subseteq (2^{\text{AP}})^\omega$  we define:

$$\begin{aligned} T \models^* p & \quad \text{if for all } t \in T: p \in t(0), \\ T \models^* \neg p & \quad \text{if for all } t \in T: p \notin t(0), \\ T \models^* \psi \wedge \varphi & \quad \text{if } T \models^* \psi \text{ and } T \models^* \varphi, \\ T \models^* \psi \vee \varphi & \quad \text{if there are } T_1 \cup T_2 = T \text{ such that } T_1 \models^* \psi \text{ and } T_2 \models^* \varphi, \text{ and} \\ T \models^* \text{X}\varphi & \quad \text{if } T[1, \infty) \models^* \varphi. \end{aligned}$$

<sup>2</sup>The length of an LTL formula is often defined to be the number of syntactically different subformulas, which might be exponentially smaller. Here, we need to distinguish syntactically equal subformulas.

This concludes the cases where both semantics coincide. Next, we present the remaining cases for the synchronous semantics, which are inherited from the classical semantics.

$$\begin{aligned}
T \models^s F\varphi & \text{ if there is a } k \geq 0 \text{ such that } T[k, \infty) \models^s \varphi, \\
T \models^s G\varphi & \text{ if for all } k \geq 0: T[k, \infty) \models^s \varphi, \\
T \models^s \psi U \varphi & \text{ if there is a } k \geq 0 \text{ such that } T[k, \infty) \models^s \varphi \text{ and for all } k' < k: T[k', \infty) \models^s \psi, \text{ and} \\
T \models^s \psi R \varphi & \text{ if for all } k \geq 0: T[k, \infty) \models^s \varphi \text{ or there is a } k' < k \text{ such that } T[k', \infty) \models^s \psi.
\end{aligned}$$

Finally, we present the remaining cases for the asynchronous semantics. Here, there is not a unique timepoint  $k$ , but a timepoint  $k_t$  for every trace  $t$ , i.e., time evolves asynchronously between different traces.

$$\begin{aligned}
T \models^a F\varphi & \text{ if for all } t \in T \text{ there is a } k_t \geq 0 \text{ such that } \{t[k_t, \infty) \mid t \in T\} \models^a \varphi, \\
T \models^a G\varphi & \text{ if for all } t \in T \text{ and for all } k_t \geq 0: \{t[k_t, \infty) \mid t \in T\} \models^a \varphi, \\
T \models^a \psi U \varphi & \text{ if for all } t \in T \text{ there is a } k_t \geq 0 \text{ such that } \{t[k_t, \infty) \mid t \in T\} \models^a \varphi \text{ and} \\
& \text{for all } k'_t < k_t: \{t[k'_t, \infty) \mid t \in T\} \models^a \psi, \text{ and} \\
T \models^a \psi R \varphi & \text{ if for all } t \in T \text{ and for all } k_t \geq 0: \{t[k_t, \infty) \mid t \in T\} \models^a \varphi \text{ or} \\
& \text{there is a } k'_t \leq k_t \text{ such that: } \{t[k'_t, \infty) \mid t \in T\} \models^a \psi.
\end{aligned}$$

We call expressions of the form  $\psi \vee \varphi$  *splitjunctions* to emphasise on the team semantics where we split a team into two parts. Similarly, the  $\vee$ -operator is referred to as *splitjunction*.

Let us illustrate the difference between synchronous and asynchronous semantics with an example involving the eventually operator. Similar examples can be constructed for the other temporal operators (but for  $X$ ) as well.

**Example 1** *Let  $T = \{\{p\}\emptyset^\omega, \emptyset\{p\}\emptyset^\omega\}$ . Then, we have  $T \models^a Fp$ , as we can pick  $k_t = 0$  if  $t = \{p\}\emptyset^\omega$ , and can pick  $k_t = 1$  if  $t = \emptyset\{p\}\emptyset^\omega$ . On the other hand, there is no single  $k$  such that  $T[k, \infty) \models^s p$ , as the occurrences of  $p$  are at different positions. Concluding,  $T \not\models^s Fp$ .*

On the other hand, synchronous satisfaction implies asynchronous satisfaction, i.e.,  $T \models^s \varphi$  implies  $T \models^a \varphi$ . The simplest way to prove this is by applying downward closure, singleton equivalence, and flatness (see below). Example 1 shows that the converse does not hold, i.e., asynchronous satisfaction does not imply synchronous satisfaction.

### 3 Basic Properties of Team Semantics

Here, we consider several standard properties of team semantics (cf., e.g. [10]) and verify which hold for our two semantics for LTL. These properties are later used to analyse the complexity of the satisfiability and model checking problems. To simplify our notation,  $\models^*$  again stands for  $\models^a$  or  $\models^s$ . We say that a semantics  $\models^*$

- satisfies the *empty team property*, if every formula is satisfied by the empty team, i.e., if  $\emptyset \models^* \varphi$  for every LTL formula  $\varphi$ ,
- is *downwards closed*, if satisfying teams are closed under taking subteams, i.e., if for every team  $T \subseteq (2^{AP})^\omega$ ,  $T \models^* \varphi$  implies  $T' \models^* \varphi$  for all  $T' \subseteq T$ ,
- satisfies *singleton equivalence*, if for every trace  $t \in (2^{AP})^\omega$ ,  $\{t\} \models^* \varphi$  if and only if  $t \models^c \varphi$ ,
- is *union closed*, if for all teams  $T, T' \subseteq (2^{AP})^\omega$ :  $T \models^* \varphi$  and  $T' \models^* \varphi$  implies  $T \cup T' \models^* \varphi$ ,
- is *flat*, if for every team  $T \subseteq (2^{AP})^\omega$ :  $T \models^* \varphi$  if and only if  $t \models^c \varphi$  for all  $t \in T$ .

## Lemma 2

1. *Asynchronous team semantics satisfies all five properties above.*
2. *Synchronous team semantics satisfies the empty team property, is downwards closed, and satisfies singleton equivalence, but is neither union closed nor flat.*

**PROOF Empty team property:** By induction over the construction of  $\varphi$ .

**Downward closure:** By induction over the construction of  $\varphi$ . The induction start for formulas  $p$  and  $\neg p$  relies on the semantics for these subformulas universally quantifying over the traces in  $T$ . All other cases are straightforward.

**Singleton equivalence:** Again, by an induction over the construction of  $\varphi$  showing that both semantics boil down to classical semantics for singletons.

**Union closure:** First consider  $\models^a$ . Here, union closure can be shown by an induction over the construction of  $\varphi$ . In the induction step for the temporal operators  $F$ ,  $G$ ,  $U$ , and  $R$  we rely on asynchronicity, as we can pick a different  $k_t$  for every trace  $t$ .

Now, consider  $\models^s$  and let  $T = \{\{p\}\emptyset^\omega\}$  and  $T' = \{\emptyset\{p\}\emptyset^\omega\}$ . Then, we have  $T \models^s Fp$  and  $T' \models^s Fp$  by singleton equivalence, but  $T \cup T' \not\models^s Fp$  as argued above.

**Flatness:** To prove flatness for  $\models^a$ , apply union closure and singleton equivalence from left to right (which also holds for uncountable unions) and downward closure and singleton equivalence from right to left.

For  $\models^s$ , the counterexample from the proof of union closure is applicable here as well. ■

Figure 1 summarises the proven structural properties of the investigated semantics.

property	definition	$\models^a$	$\models^s$
empty team property	$\emptyset \models^* \varphi$	✓	✓
downwards closure	$T \models^* \varphi$ implies $\forall T' \subseteq T: T' \models^* \varphi$	✓	✓
union closure	$T \models^* \varphi, T' \models^* \varphi$ implies $T \cup T' \models^* \varphi$	✓	×
flatness	$T \models^* \varphi$ if and only if $\forall t \in T: t \models^c \varphi$	✓	×
singleton equivalence	$\{t\} \models^* \varphi$ if and only if $t \models^c \varphi$	✓	✓

Figure 1: Structural properties overview.

## 4 Complexity Results for LTL under Team Semantics

In this section, we determine the complexity of the most important verification problems for LTL under team semantics, namely satisfiability (Subsection 4.1) and two variations of the model checking problem: For classical LTL, one studies the path checking problem and the model checking problem. The difference between these two problems lies in the type of structures one considers. Recall that a model of an LTL formula is a single trace. In the path checking problem, one is given such a trace  $t$  and a formula  $\varphi$ , and has to decide  $t \models^c \varphi$ . This problem has applications to runtime verification and monitoring of reactive systems [25, 28]. In the model checking problem, one is given a Kripke structure  $\mathcal{K}$  and a formula  $\varphi$ , and has to decide whether every execution trace  $t$  of  $\mathcal{K}$  satisfies  $\varphi$ . We study path checking in Subsection 4.2 and model checking in Subsection 4.3.

## 4.1 Satisfiability

In this subsection, we show that satisfiability under (synchronous and asynchronous) team semantics boils down to satisfiability under classical semantics, which is well-understood (see, e.g., [8, 33] and the references in the former paper). Formally, we are interested in the following problem.

**Problem:** Team LTL Satisfiability (TSAT $^*$ ) for  $\star \in \{a, s\}$

**Input:** LTL formula  $\varphi$

**Question:** Is there a non-empty team  $T$  such that  $T \models^{\star} \varphi$ ?

Note that the non-emptiness condition is necessary, as otherwise every formula is satisfiable due to the empty team property. We show that satisfiability for both team semantics is as hard as the satisfiability problem for LTL under classical semantics (denoted by LTL-SAT).

**Theorem 3** TSAT $^a \equiv_{cd}$  LTL-SAT and TSAT $^s \equiv_{cd}$  LTL-SAT.

**PROOF** Let  $\star \in \{a, s\}$  and let  $\varphi$  be an LTL formula. We show that there is a non-empty team  $T$  with  $T \models^{\star} \varphi$  if and only if there is a trace  $t$  with  $t \models^c \varphi$ , which yields the desired results by identity.

Consequently, let  $T \models^{\star} \varphi$  for some non-empty team  $T$ . We fix some  $t \in T$ . By downward closure, we obtain  $\{t\} \models^{\star} \varphi$  and then by the singleton equivalence  $t \models^c \varphi$ .

For the converse implication, let  $t \models^c \varphi$ . Then, by singleton equivalence,  $\{t\} \models^{\star} \varphi$ .

**Proposition 4** ([33]) LTL satisfiability is PSPACE-complete w.r.t.  $\leq_m^P$ -reductions.

Together, we can immediately deduce the following corollary.

**Corollary 5** TSAT $^a$  and TSAT $^s$  are both PSPACE-complete w.r.t.  $\leq_m^P$ -reductions.

## 4.2 Path Checking

In this subsection, we consider a generalisation of the path checking problem for LTL (denoted by LTL-PC), which asks for a given ultimately periodic trace  $t$  and a given formula  $\varphi$ , whether  $t \models^c \varphi$  holds. Here, we investigate the complexity of checking whether a given finite team comprised of ultimately periodic traces satisfies a given formula. Such a team is given by a team encoding  $\mathcal{T}$ . To simplify our notation, we will write  $\mathcal{T} \models^{\star} \varphi$  instead of  $\llbracket \mathcal{T} \rrbracket \models^{\star} \varphi$ .

**Problem:** TeamPathChecking (TPC $^*$ ) for  $\star \in \{a, s\}$

**Input:** LTL formula  $\varphi$  and a finite team encoding  $\mathcal{T}$ .

**Question:**  $\mathcal{T} \models^{\star} \varphi$ ?

Our main results settle the complexity of TPC $^*$  for  $\star \in \{a, s\}$ . We begin by considering asynchronous semantics.

**Theorem 6** TPC $^a \equiv_{cd}$  LTL-PC.

**PROOF** We have  $\langle \varphi, \mathcal{T} \rangle \in \text{TPC}^a$  if and only if  $\bigwedge_{(t_0, t_1) \in \mathcal{T}} (\langle \varphi, (t_0, t_1) \rangle \in \text{LTL-PC})$ . This can be computed by an AC $^0$ -circuit using oracle gates for LTL-PC.

Now,  $\langle \varphi, (t_0, t_1) \rangle \in \text{LTL-PC}$  if and only if  $\langle \varphi, \{(t_0, t_1)\} \rangle \in \text{TPC}^a$  yields the other direction.

The exact complexity of LTL-PC is an open problem; the best bounds on the problem are as follows:

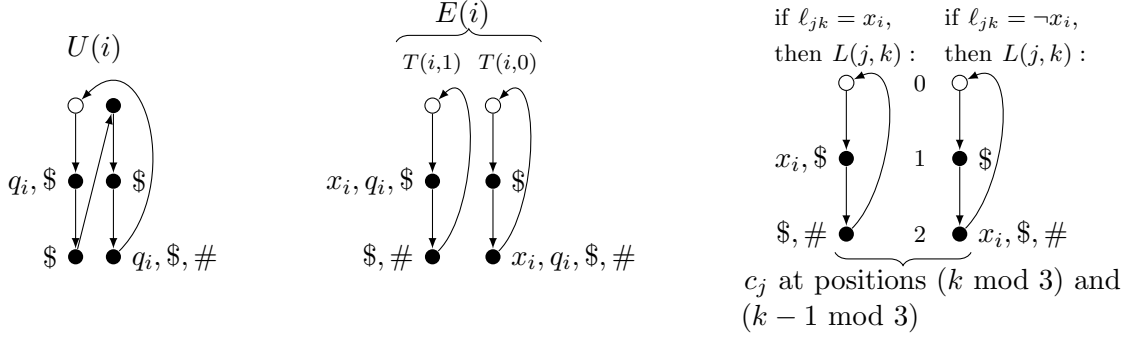


Figure 2: Traces for the reduction defined in the proof of Lemma 9.

**Proposition 7** ([25, 28], see also [24]) *LTL-PC is in  $AC^1(\log DCFL)$  and  $NC^1$ -hard w.r.t.  $\leq_{cd}$ -reductions.*

Combining the previous two results we deduce the following corollary.

**Corollary 8** *TPC<sup>a</sup> is in  $AC^1(\log DCFL)$  and  $NC^1$ -hard w.r.t.  $\leq_{cd}$ -reductions.*

Now, we turn towards the path checking problem for the synchronous semantics. It turns out that this problem is harder than the asynchronous version.

**Lemma 9** *TPC<sup>s</sup> is PSPACE-hard w.r.t.  $\leq_m^P$ -reductions.*

**PROOF** Determining whether a given quantified Boolean formula (qBf) is valid (QBF-VAL) is a well-known PSPACE-complete problem [27]. The problem stays PSPACE-complete if the matrix of the given gBf is in 3CNF. To prove the claim of the lemma, we will show that QBF-VAL  $\leq_m^P$  TPC<sup>s</sup>. Given a quantified Boolean formula  $\varphi$ , we stipulate, w.l.o.g., that  $\varphi$  is of the form  $\exists x_1 \forall x_2 \cdots Q x_n \chi$ , where  $\chi = \bigwedge_{j=1}^m \bigvee_{k=1}^3 \ell_{jk}$ ,  $Q \in \{\exists, \forall\}$ , and  $x_1, \dots, x_n$  are exactly the free variables of  $\chi$  and pairwise distinct.

In the following we define a reduction which is composed of two functions  $f$  and  $g$ . Given a qBf  $\varphi$ , the function  $f$  will define an LTL-formula and  $g$  will define a team such that  $\varphi$  is valid if and only if  $g(\varphi) \models f(\varphi)$ . Essentially, the team  $g(\varphi)$  will contain three kinds of traces: (i) traces which are used to mimic universal quantification ( $U(i)$  and  $E(i)$  below), (ii) traces that are used to simulate existential quantification ( $E(i)$  below), and (iii) traces used to encode the matrix of  $\varphi$  ( $L(j, k)$  below). Moreover the trace  $T(i, 1)$  ( $T(i, 0)$ , resp.) is used inside the proof to encode an assignment that maps the variable  $x_i$  true (false, resp.). Note that,  $U(i), T(i, 1), T(i, 0), L(j, k)$  technically are singleton sets of traces. For convenience, we identify them with the traces they contain.

Next we inductively define the reduction function  $f$  that maps qBf-formulas to LTL-formulas:

$$f(\chi) := \bigvee_{i=1}^n Fx_i \vee \bigvee_{i=1}^m Fc_i,$$

where  $\chi$  is the 3CNF-formula  $\bigwedge_{j=1}^m \bigvee_{k=1}^3 \ell_{jk}$  with free variables  $x_1, \dots, x_n$ ,

$$\begin{aligned} f(\exists x_i \psi) &:= (Fq_i) \vee f(\psi), \\ f(\forall x_i \psi) &:= (\$ \vee (\neg q_i Uq_i) \vee F[\# \wedge Xf(\psi)]) U\#. \end{aligned}$$



The reduction function  $g$  that maps qBf-formulas to teams is defined as follows with respect to the traces in Figure 2.

$$g(\chi) := \bigcup_{j=1}^m L(j, 1) \cup L(j, 2) \cup L(j, 3),$$

where  $\chi$  is the 3CNF-formula  $\bigwedge_{j=1}^m \bigvee_{k=1}^3 \ell_{jk}$  with free variables  $x_1, \dots, x_n$  and

$$\begin{aligned} g(\exists x_i \psi) &:= E(i) \cup g(\psi), \\ g(\forall x_i \psi) &:= U(i) \cup E(i) \cup g(\psi). \end{aligned}$$

The first position of each trace is marked with a white circle. For instance, the trace of  $U(i)$  then is encoded via  $(\varepsilon, \emptyset\{q_i, \$\}\{\$\}\emptyset\{\$\}\{q_i, \#, \$\})$ . The reduction function showing  $\text{QBF-VAL} \leq_m^p \text{TPC}^s$  is then  $\varphi \mapsto \langle g(\varphi), f(\varphi) \rangle$ . Clearly  $f(\varphi)$  and  $g(\varphi)$  can be computed in linear time with respect to  $|\varphi|$ .

Intuitively, for the existential case, the formula  $(Fq_i) \vee f(\psi)$  allows to continue in  $f(\psi)$  with exactly one of  $T(i, 1)$  or  $T(i, 0)$ . If  $b \in \{0, 1\}$  is a truth value then selecting  $T(i, b)$  in the team is the same as setting  $x_i$  to  $b$ . For the case of  $f(\forall x_i \psi)$ , the formula  $(\neg q_i \cup q_i) \vee F[\#\ \wedge \mathbf{X}f(\psi)]$  with respect to the team  $(U(i) \cup E(i))[0, \infty)$  is similar to the existential case choosing  $x_i$  to be 1 whereas for  $(U(i) \cup E(i))[3, \infty)$  one selects  $x_i$  to be 0. The use of the until operator in combination with  $\$$  and  $\#$  then forces both cases to happen.

Let  $\varphi' = Q'x_{n'+1} \cdots Q_n \chi$ , where  $Q', Q \in \{\exists, \forall\}$  and let  $I$  be an assignment of the variables in  $\{x_1, \dots, x_{n'}\}$  for  $n' \leq n$ . Then, we define

$$g(I, \varphi') := g(\varphi') \cup \bigcup_{x_i \in \text{Dom}(I)} T(i, I(x_i)).$$

We claim  $I \models \varphi'$  if and only if  $g(I, \varphi') \models^s f(\varphi')$ . Note that when  $\varphi' = \varphi$  it follows that  $I = \emptyset$  and that  $g(I, \varphi') = g(\varphi)$ . Accordingly, the lemma follows from the claim of correctness. We prove the claim by induction on the number of quantifier alternations in  $\varphi'$ .

**IB:**  $\varphi' = \chi$ , this implies that  $\varphi'$  is quantifier-free and  $\text{Dom}(I) = \{x_1, \dots, x_n\}$ .

“ $\Leftarrow$ ”: Let  $g(I, \varphi') = T_1 \cup T_2$  such that  $T_1 \models^s \bigvee_{i=1}^n Fx_i$  and  $T_2 \models^s \bigvee_{i=1}^n Fc_i$ . We assume w.l.o.g.  $T_1$  and  $T_2$  to be disjoint, which is possible due to downwards closure. Note that then we have  $T_2 \subseteq \{L(j, k) \mid 1 \leq j \leq m, 1 \leq k \leq 3\}$  and  $T_1 = (\{L(j, k) \mid 1 \leq j \leq m, 1 \leq k \leq 3\} \setminus T_2) \cup \{T(i, I(x_i)) \mid 1 \leq i \leq n\}$ , as for any  $1 \leq i, i' \leq m$ ,  $c_i$  does not appear positively in the traces in  $T(i', I(x_{i'}))$ . Observe that, for any clause  $1 \leq j \leq m$ , any two of  $L(j, k)$ , for  $1 \leq k \leq 3$ , (but not all three) can belong to  $T_2$ . Due to construction of the traces,  $L(j, k) \in T_2$  can only satisfy the subformula  $Fc_j$  and not other  $Fc_{j'}$  with  $j \neq j'$ . The reason is that there exists no  $s \in \mathbb{N}$  such that  $L(j, k)(s) \ni c_j$  for all  $1 \leq k \leq 3$  and consequently  $Fc_j$  is falsified. As a result,

$$T_2 \not\supseteq \{L(j, 1), L(j, 2), L(j, 3)\} \text{ for each } 1 \leq j \leq m.$$

As  $T_1 \supseteq \{T(i, I(x_i)) \mid 1 \leq i \leq n\}$ , for each  $1 \leq i \leq n$  we need to divide  $T_1 = T_1^1 \cup \dots \cup T_1^n$  such that  $T_1^i \models^s Fx_i$  as only  $Fx_i$  can be satisfied by  $T(i, I(x_i))$  and by no other  $T(i', I(x_{i'}))$  (for a different  $i' \neq i$ ). Note that, if  $L(j, k) \in T_1$  it has to be in  $T_1^i$  where  $x_i$  is the variable of  $\ell_{j,k}$ . By construction of the traces, if  $T(i, 1) \in T_1^i$  we have  $T_1^i(1) \models^s x_i$  and if  $T(i, 0) \in T_1^i$  then  $T_1^i(2) \models^s x_i$ . As a result,  $I \models \ell_{j,k}$ . Further, for each  $1 \leq j \leq m$  there is a  $1 \leq k \leq 3$  such that  $L(j, k) \in T_1$  and then there exist an  $1 \leq i \leq n$  with  $T_1^i \models^s L(j, k)$  and also  $I \models \ell_{jk}$ . Consequently  $I \models \varphi'$ .

“ $\Rightarrow$ ”: Now assume that  $I \models \varphi'$ . As a result, pick for each  $1 \leq j \leq m$  a *single*  $1 \leq k \leq 3$  such that  $I \models \ell_{jk}$ . Denote this sequence of choices by  $k_1, \dots, k_m$ . Choose  $g(I, \varphi') = T_1 \cup T_2$  as follows:

$$\begin{aligned} T_1 &:= \{L(j, k_j) \mid 1 \leq j \leq m\} \cup \{T(i, I(x_i)) \mid 1 \leq i \leq n\} \\ T_2 &:= \{L(j, 1), L(j, 2), L(j, 3) \mid 1 \leq j \leq m\} \setminus T_1 \end{aligned}$$

Then,  $T_2 \models^s \bigvee_{j=1}^m Fc_j$  as exactly two traces per clause are in  $T_2$  and we can divide  $T_2 = T_2^1 \cup \dots \cup T_2^m$  where

$$T_2^j := \{L(j, k), L(j, k') \mid k, k' \in \{1, 2, 3\} \setminus \{k_j\}\}.$$

Then  $T_2^j \models^s Fc_j$  for all  $1 \leq j \leq m$  as follows.

- If  $\{L(j, 1), L(j, 2)\} = T_2$  then  $c_j \in L(j, k)(1)$  for  $k = 1, 2$ .
- If  $\{L(j, 1), L(j, 3)\} = T_2$  then  $c_j \in L(j, k)(0)$  for  $k = 1, 3$ .
- If  $\{L(j, 2), L(j, 3)\} = T_2$  then  $c_j \in L(j, k)(2)$  for  $k = 2, 3$ .

Further,  $T_1 \models^s \bigvee_{i=1}^n Fx_i$  as we can divide  $T_1 := T_1^1 \cup \dots \cup T_1^n$  such that

$$T_1^i = \{L(j, k_j) \mid 1 \leq j \leq m, I(x_i) \models \ell_{jk}\} \cup \{T(i, I(x_i))\}.$$

Then there are two possibilities:

- $I(x_i) = 1$ : then  $x_i \in (L(j, k_j)(1) \cap T(i, I(x_i))(1))$ .
- $I(x_i) = 0$ : then  $x_i \in (L(j, k_j)(2) \cap T(i, I(x_i))(2))$ .

As a result, in both cases,  $T_1^i \models^s Fx_i$ .

That being so,  $g(I, \varphi') \models^s f(\varphi')$  and the induction basis is proven.

**IS:** “Case  $\varphi' = \exists x_i \psi$ .” We need to show that  $I \models \exists x_i \psi$  if and only if  $g(I, \exists x_i \psi) \models^s f(\exists x_i \psi)$ .

First note that  $g(I, \exists x_i \psi) \models^s f(\exists x_i \psi)$  if and only if  $E(i) \cup g(\psi) \models^s (Fq_i) \vee f(\psi)$ , by the definitions of  $f$  and  $g$ . Clearly,  $E(i) \not\models^s Fq_i$ , but both  $T(i, 1) \models^s Fq_i$  and  $T(i, 0) \models^s Fq_i$ . Observe that  $E(i) = \{T(i, 1), T(i, 0)\}$  and  $q_i$  does not appear positively anywhere in  $g(\psi)$ . Accordingly, and by downwards closure,  $E(i) \cup g(\psi) \models^s (Fq_i) \vee f(\psi)$  if and only if

$$\exists b \in \{0, 1\} \text{ s.t. } T(i, 1 - b) \models^s Fq_i \text{ and } (E(i) \cup g(\psi)) \setminus T(i, 1 - b) \models^s f(\psi). \quad (1)$$

Since  $(E(i) \cup g(\psi)) \setminus T(i, 1 - b) = T(i, b) \cup g(\psi) = g(I[x_i \mapsto b], \psi)$ , Equation (1) holds if and only if  $g(I[x_i \mapsto b], \psi) \models^s f(\psi)$ , for some bit  $b \in \{0, 1\}$ . By the induction hypothesis, the latter holds if and only if there exists a bit  $b \in \{0, 1\}$  s.t.  $I[x_i \mapsto b] \models \psi$ . Finally by the semantics of  $\exists$  this holds if and only if  $I \models \exists x_i \psi$ .

“Case  $\varphi' = \forall x_i \psi$ .” We need to show that  $I \models \forall x_i \psi$  if and only if  $g(I, \forall x_i \psi) \models^s f(\forall x_i \psi)$ .

First note that, by the definitions of  $f$  and  $g$ ,  $g(I, \forall x_i \psi) \models^s f(\forall x_i \psi)$  if and only if

$$U(i) \cup E(i) \cup g(\psi) \models^s (\$ \vee (\neg q_i \cup q_i) \vee F[\# \wedge Xf(\psi)]) \cup \#. \quad (2)$$

We claim that (2) holds if and only if  $T(i, b) \cup g(\psi) \models^s f(\psi)$  for all  $b \in \{0, 1\}$ . From this the correctness follows analogously as in the case for the existential quantifier.

Proof of the claim: Notice first that each trace in  $U(i) \cup E(i) \cup g(\psi)$  is periodic with period length either 3 or 6, and exactly the last element of each period is marked by the symbol  $\#$ . Consequently, it is easy to see that (2) holds if and only if

$$(U(i) \cup E(i) \cup g(\psi))[j, \infty) \models^s \$ \vee (\neg q_i \cup q_i) \vee F[\# \wedge Xf(\psi)], \text{ for each } j \in \{0, 1, 2, 3, 4\}. \quad (3)$$

Note that  $(U(i) \cup E(i) \cup g(\psi))[j, \infty) \models \$$ , for each  $j \in \{1, 2, 4\}$ , whereas no non-empty subteam of  $(U(i) \cup E(i) \cup g(\psi))[j, \infty)$ ,  $j \in \{0, 3\}$  satisfies  $\$$ . Accordingly, (3) holds if and only if

$$(U(i) \cup E(i) \cup g(\psi))[j, \infty) \models (\neg q_i \mathbf{U} q_i) \vee \mathbf{F}[\# \wedge \mathbf{X}f(\psi)], \text{ for both } j \in \{0, 3\}. \quad (4)$$

Note that, by construction,  $q_i$  does not occur positively in  $g(\psi)$ . As a result,  $X \cap g(\psi)[j, \infty) = \emptyset$ ,  $j \in \{0, 3\}$ , for all teams  $X$  s.t.  $X \models \neg q_i \mathbf{U} q_i$ . Also, none of the symbols  $x_{i'}$ ,  $c_{i'}$ ,  $q_{i''}$ , for  $i', i'' \in \mathbb{N}$  with  $i'' \neq i$ , occurs positively in  $U(i)$ . On that account,  $X \cap U(i)[j, \infty) = \emptyset$ ,  $j \in \{0, 3\}$ , for all  $X$  s.t.  $X \models \mathbf{F}[\# \wedge \mathbf{X}f(\psi)]$ , for eventually each trace in  $X$  will end up in a team that satisfies one of the formulas of the form  $\mathbf{F}x_{i'}$ ,  $\mathbf{F}c_{i'}$ , or  $\mathbf{F}q_{i''}$  (see the inductive definition of  $f$ ). Moreover, it is easy to check that  $(T(i, 1) \cup U(i))[0, \infty) \models \neg q_i \mathbf{U} q_i$ ,  $(T(i, 0) \cup U(i))[0, \infty) \not\models \neg q_i \mathbf{U} q_i$ ,  $(T(i, 0) \cup U(i))[3, \infty) \models \neg q_i \mathbf{U} q_i$ , and  $(T(i, 1) \cup U(i))[3, \infty) \not\models \neg q_i \mathbf{U} q_i$ . From these, together with downwards closure, it follows that (4) holds if and only if for  $b_0 = 1$  and  $b_3 = 0$

$$(U(i) \cup T(i, b_j))[j, \infty) \models \neg q_i \mathbf{U} q_i, \text{ for all } j \in \{0, 3\} \quad (5)$$

and

$$(T(i, 1 - b_j) \cup g(\psi))[j, \infty) \models \mathbf{F}[\# \wedge \mathbf{X}f(\psi)], \text{ for both } j \in \{0, 3\}. \quad (6)$$

In fact, as (5) always holds, (4) is equivalent with (6). By construction, (6) holds if and only if  $(T(i, b) \cup g(\psi))[6, \infty) \models f(\psi)$ , for both  $b \in \{0, 1\}$ . Now, since  $(T(i, b) \cup g(\psi))[6, \infty) = T(i, b) \cup g(\psi)$  the claim applies.  $\blacksquare$

Now we turn our attention to proving a matching upper bound. To this end, we need to introduce some notation to manipulate team encodings. Given a pair  $(t_0, t_1)$  of traces  $t_0 = t_0(0) \cdots t_0(n)$  and  $t_1 = t_1(0) \cdots t_1(n')$ , we define  $(t_0, t_1)[1, \infty)$  to be  $(t_0(1) \cdots t_0(n), t_1)$  if  $t_0 \neq \varepsilon$ , and to be  $(\varepsilon, t_1(1) \cdots t_1(n')t_1(0))$  if  $t_0 = \varepsilon$ . Furthermore, we inductively define  $(t_0, t_1)[i, \infty)$  to be  $(t_0, t_1)$  if  $i = 0$ , and to be  $((t_0, t_1)[1, \infty))[i - 1, \infty)$  if  $i > 0$ . Then,  $\llbracket (t_0, t_1)[i, \infty) \rrbracket = (\llbracket (t_0, t_1) \rrbracket)[i, \infty)$ , i.e., we have implemented the prefix-removal operation on the finite representation. Furthermore, we lift this operation to team encodings  $\mathcal{T}$  by defining  $\mathcal{T}[i, \infty) = \{(t_0, t_1)[i, \infty) \mid (t_0, t_1) \in \mathcal{T}\}$ . As a result, we have  $\llbracket \mathcal{T}[i, \infty) \rrbracket = (\llbracket \mathcal{T} \rrbracket)[i, \infty)$ .

Given a finite team encoding  $\mathcal{T}$ , let  $\text{prfx}(\mathcal{T}) = \max\{|t_0| \mid (t_0, t_1) \in \mathcal{T}\}$  and let  $\text{lcm}(\mathcal{T})$  be the least common multiple of  $\{|t_1| \mid (t_0, t_1) \in \mathcal{T}\}$ . Then,  $\mathcal{T}[i, \infty) = \mathcal{T}[i + \text{lcm}(\mathcal{T}), \infty)$  for every  $i \geq \text{prfx}(\mathcal{T})$ . The next remark is now straightforward.

**Remark 1** Let  $\mathcal{T}$  be a finite team encoding and let  $i \geq \text{prfx}(\mathcal{T})$ . Then,  $\mathcal{T}[i, \infty)$  and  $\mathcal{T}[i + \text{lcm}(\mathcal{T}), \infty)$  satisfy exactly the same LTL formulas under synchronous team semantics.  $\square$

In particular, we obtain the following consequences for temporal operators (for finite  $\mathcal{T}$ ):

- $\mathcal{T} \models \mathbf{F}\varphi$  if and only if there is a  $k \leq \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})$  such that  $\mathcal{T}[k, \infty) \models \varphi$ .
- $\mathcal{T} \models \mathbf{G}\varphi$  if and only if for all  $k \leq \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})$ :  $\mathcal{T}[k, \infty) \models \varphi$ .
- $\mathcal{T} \models \psi \mathbf{U} \varphi$  if and only if there is a  $k \leq \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})$  such that  $\mathcal{T}[k, \infty) \models \varphi$  and for all  $k' < k$ :  $\mathcal{T}[k', \infty) \models \psi$ .
- $\mathcal{T} \models \psi \mathbf{R} \varphi$  if and only if for all  $k \leq \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})$ :  $\mathcal{T}[k, \infty) \models \varphi$  or there is a  $k' < k$  such that  $\mathcal{T}[k', \infty) \models \psi$ .

Accordingly, we can restrict the range of the temporal operators when model checking a finite team encoding. This implies that a straightforward recursive algorithm implementing the synchronous semantics solves LTL-PC<sup>s</sup>.

**Lemma 10**  $\text{TPC}^s$  is in PSPACE.

PROOF Consider Algorithm 1. Note that when combining the results of recursive calls,  $\vee$  and  $\bigvee$  denote classical disjunction, not splitjunction.

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**Algorithm 1:** Path checking algorithm for LTL under synchronous team semantics

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1 Procedure p-check(Team encoding  $\mathcal{T}$ , formula  $\varphi$ );
2 if  $\varphi = p$  then return  $\bigwedge_{(t_0, t_1) \in \mathcal{T}} p \in t_0 t_1(0)$ ;
3 if  $\varphi = \neg p$  then return  $\bigwedge_{(t_0, t_1) \in \mathcal{T}} p \notin t_0 t_1(0)$ ;
4 if  $\varphi = \psi \wedge \psi'$  then return p-check( $\mathcal{T}, \psi$ )  $\wedge$  p-check( $\mathcal{T}, \psi'$ );
5 if  $\varphi = \psi \vee \psi'$  then return  $\bigvee_{\mathcal{T}' \subseteq \mathcal{T}} \text{p-check}(\mathcal{T}', \psi) \wedge \text{p-check}(\mathcal{T} \setminus \mathcal{T}', \psi')$ ;
6 if  $\varphi = X\psi$  then return p-check( $\mathcal{T}[1, \infty), \psi$ );
7 if  $\varphi = F\psi$  then return  $\bigvee_{k \leq \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})} \text{p-check}(\mathcal{T}[k, \infty), \psi)$ ;
8 if  $\varphi = G\psi$  then return  $\bigwedge_{k \leq \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})} \text{p-check}(\mathcal{T}[k, \infty), \psi)$ ;
9 if  $\varphi = \psi U \psi'$  then return (  $\bigvee_{k \leq \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})} \text{p-check}(\mathcal{T}[k, \infty), \psi') \wedge$ 
    $\bigwedge_{k' < k} \text{p-check}(\mathcal{T}[k', \infty), \psi)$  );
10 if  $\varphi = \psi R \psi'$  then return (  $\bigwedge_{k \leq \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})} \text{p-check}(\mathcal{T}[k, \infty), \psi') \vee$ 
    $\bigvee_{k' < k} \text{p-check}(\mathcal{T}[k', \infty), \psi)$  );

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The algorithm is an implementation of the synchronous team semantics for LTL with slight restrictions to obtain the desired complexity. In line 5, we only consider strict splits, i.e., the team is split into two disjoint parts. This is sufficient due to downwards closure. Furthermore, the scope of the temporal operators in lines 7 to 10 is restricted to the interval  $[0, \text{prfx}(\mathcal{T}) + \text{lcm}(\mathcal{T})]$ . This is sufficient due to the above observations.

It remains to analyse the algorithm's space complexity. Its recursion depth is bounded by the size of the formula. Further, in each recursive call, a team encoding has to be stored. Additionally, in lines 5 and 7 to 10, a disjunction or conjunction of exponential arity has to be evaluated. In each case, this only requires linear space in the input to make the recursive calls and to aggregate the return value. As a result, Algorithm 1 can be implemented in polynomial space.  $\blacksquare$

Combining Lemma 9 and Lemma 10 settles the exact complexity of  $\text{TPC}^s$ .

**Theorem 11**  $\text{TPC}^s$  is PSPACE-complete w.r.t.  $\leq_m^P$ -reductions.

It turns out that this algorithm is very robust to strengthenings of the logic under consideration. To conclude this subsection, we consider two extensions of LTL under synchronous team semantics and show that our algorithm is still applicable to them.

First, we consider the addition of more expressive atomic formulas. We are interested in atoms expressible in first-order (FO) logic over the atomic propositions; most widely studied of which are dependence, independence, and inclusion atoms [10]. The notion of generalised atoms in the setting of first-order team semantics was introduced by Kuusisto [26].

We consider FO formulas over the signature  $(A_p)_{p \in \text{AP}}$ , where each  $A_p$  is a unary predicate. Furthermore, we interpret a team  $T$  as a relational structure  $\mathfrak{A}(T)$  over the same signature with universe  $T$  such that  $t \in T$  is in  $A_p^{\mathfrak{A}}$  if and only if  $p \in t(0)$ . So, formulas can express properties of the atomic propositions holding in the initial positions of traces in  $T$ . Then, an FO formula  $\varphi$  FO-defines the atomic formula  $D$  with  $T \models^* D \iff \mathfrak{A}(T) \models \varphi$ . In this case,  $D$  is also called an *generalised atom*.

For instance, the dependence atom  $=(x, y)$  is FO-definable by  $\forall t \forall t' ((A_x(t) \leftrightarrow A_x(t')) \rightarrow (A_y(t) \leftrightarrow A_y(t')))$ , for  $x, y \in \text{AP}$ . We call an LTL formula extended by a generalised atom  $D$  also an LTL( $D$ ) formula. Similarly, we lift this notion to *sets of generalised atoms* as well as to the corresponding decision problems, i.e.,  $\text{TPC}^s(D)$  is the path checking problem over synchronous semantics with LTL formulas which may use the generalised atom  $D$ .

**Theorem 12** *Let  $\mathcal{D}$  be a finite set of generalised atoms. Then  $\text{TPC}^s(\mathcal{D})$  is PSPACE-complete w.r.t.  $\leq_m^P$ -reductions.*

PROOF The lower bound applies from Theorem 11. For the upper bound, we extend the algorithm stated in the proof of Lemma 10 for the cases of FO-definable atoms. Whenever such an atom  $D$  appears in the computation of the algorithm, we need to solve an FO model checking problem. As FO model checking is solvable in logarithmic space [22] the theorem applies.

Finally, we consider another extension. Observe that, currently, we only allow LTL formulas in negation normal form. Worth noting, arbitrary negation combined with team semantics allows for powerful constructions. For instance, the complexity of model checking variants of propositional team logic then jumps from NP- to PSPACE-completeness [29], for validity and satisfiability, even more, to AEXP[poly]-completeness (alternating exponential time with polynomially many alternations) [20]. Our algorithm for the path checking problem for synchronous team semantics can conceptually straightforwardly be extended to deal with negations as well, without at a price in terms of complexity.

Formally, negation is denoted by  $\sim$  and we define  $T \models^s \sim\varphi$ , if  $T \not\models^s \varphi$ .<sup>3</sup> Then, we consider the path checking problem for LTL with arbitrary negations under synchronous team semantics, denoted by  $\text{TPC}^s(\sim)$ .

**Theorem 13**  *$\text{TPC}^s(\sim)$  is PSPACE-complete w.r.t.  $\leq_m^P$ -reductions.*

PROOF The lower bound follows from Theorem 11, while the upper bound is obtained by adding the following line to the recursive algorithm from the proof of Lemma 10 (where  $\neg$  denotes classical negation):

**if**  $\varphi = \sim\varphi'$  **then return**  $\neg\text{p-check}(\mathcal{T}, \varphi')$  ■

### 4.3 Model Checking

In this subsection, we consider the generalised model checking problem where one checks whether the team of traces of a Kripke structure satisfies a given formula. This is the natural generalisation of the model checking problem for classical semantics.

A Kripke structure  $\mathcal{K} = (W, R, \eta, w_I)$  consists of a finite set  $W$  of worlds, a left-total transition relation  $R \subseteq W \times W$ , a labeling function  $\eta: W \rightarrow 2^{\text{AP}}$ , and an initial world  $w_I \in W$ . A path  $\pi$  through  $\mathcal{K}$  is an infinite sequence  $\pi = \pi(0)\pi(1)\pi(2)\dots \in W^\omega$  such that  $\pi(0) = w_I$  and  $(\pi(i), \pi(i+1)) \in R$  for every  $i \geq 0$ . The trace of  $\pi$  is defined as  $t(\pi) = \eta(\pi(0))\eta(\pi(1))\eta(\pi(2))\dots \in (2^{\text{AP}})^\omega$ . The Kripke structure  $\mathcal{K}$  induces the team  $T(\mathcal{K}) = \{t(\pi) \mid \pi \text{ is a path through } \mathcal{K}\}$ .

**Problem:** TeamModelChecking (TMC $^\star$ ) for  $\star \in \{a, s\}$

**Input:** LTL formula  $\varphi$  and a Kripke structure  $\mathcal{K}$

**Question:**  $T(\mathcal{K}) \models^\star \varphi$ ?

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<sup>3</sup>Note that negation as defined here is not equivalent to negation of atomic propositions, i.e.,  $\sim p$  and  $\neg p$  are not equivalent due to the universal definition of the semantics of  $\neg p$ .

First, we again consider the model checking problem under asynchronous semantics, which boils down to classical LTL model checking, whose complexity is well-understood (see, e.g., the work of Baier and Katoen [1], for an overview with numerous references to original work). This problem, denoted by LTL-MC, asks, for a given Kripke structure  $\mathcal{K}$  and a given LTL formula  $\varphi$ , whether  $t \models \varphi$  for every  $t \in T(\mathcal{K})$ .

**Theorem 14**  $\text{TMC}^a \equiv_{\text{cd}} \text{LTL-MC}$ .

PROOF We have  $T(\mathcal{K}) \models^a \varphi$  if and only if  $t \models^c \varphi$  for all  $t \in T(\mathcal{K})$ , due to flatness. Accordingly, the desired reduction is trivial.

The complexity of the LTL model checking problem is well-known.

**Proposition 15** ([33]) *LTL-MC is PSPACE-complete w.r.t.  $\leq_m^P$ -reductions.*

In view of this, we obtain the following result on the complexity of the LTL model checking problem under asynchronous team semantics.

**Corollary 16**  $\text{TMC}^a$  *is PSPACE-complete w.r.t.  $\leq_m^P$ -reductions.*

Now, we consider the model checking problem under synchronous semantics. A natural approach to this problem is to generalise the algorithm for the simpler path checking problem. There, we first guessed a splitting function that implements the splits for all splitjunctions in the formula. Then, we computed the satisfaction summary for this splitting function, which encodes the result.

To lift this to the model checking problem, we have to overcome two major obstacles: How to represent teams obtained by splits and how to compute the satisfaction summary for this representation. Note that the semantics of the splitjunction ranges over arbitrary splits  $T_1 \cup T_2$  of a team  $T$ . Consequently, even if  $T = T(\mathcal{K})$  for some Kripke structure  $\mathcal{K}$ , it is possible that  $T_1$  and  $T_2$  are not representable by some Kripke structure.

In the remainder of this section, we present a model checking algorithm for the splitjunction-free fragment of LTL under synchronous team semantics. This algorithm, in conjunction with the splits always being representable by Kripke structures of bounded size, would imply decidability of the model checking problem for the full logic.

**Theorem 17**  $\text{TMC}^s$  *for the splitjunction-free fragment is in PSPACE.*

PROOF Fix  $\mathcal{K} = (W, R, \eta, w_I)$  and a splitjunction-free  $\varphi$ . We define  $S_0 = \{w_I\}$  and  $S_{i+1} = \{w' \in W \mid (w, w') \in R \text{ for some } w \in S_i\}$  for all  $i \geq 0$ . By the pigeonhole principle, this sequence is ultimately periodic with a characteristic  $(s, p)$  with  $s + p \leq 2^{|W|}$ . Next, we define a trace  $t$  over  $\text{AP} \cup \{\bar{p} \mid p \in \text{AP}\}$  via

$$t(i) = \{p \in \text{AP} \mid p \in \eta(w) \text{ for all } w \in S_i\} \cup \{\bar{p} \mid p \notin \eta(w) \text{ for all } w \in S_i\}$$

that reflects the team semantics of (negated) atomic formulas, which have to hold in every element of the team.

An induction over the construction of  $\varphi$  shows that  $T(\mathcal{K}) \models^s \varphi$  if and only if  $t \models^c \bar{\varphi}$ , where  $\bar{\varphi}$  is obtained from  $\varphi$  by replacing each negated atomic proposition  $\neg p$  by  $\bar{p}$ . To conclude the proof, we show that  $t \models^c \bar{\varphi}$  can be checked in non-deterministic polynomial space, exploiting the fact that  $t$  is ultimately periodic and of the same characteristic as  $S_0 S_1 S_2 \dots$ . However, as  $s + p$  might be exponential, we cannot just construct a finite representation of  $t$  of characteristic  $(s, p)$  and then check satisfaction in polynomial space.

Instead, we present an on-the-fly approach which is inspired by similar algorithms in the literature. It is based on the following properties:

1. Every  $S_i$  can be represented in polynomial space, and from  $S_i$  one can compute  $S_{i+1}$  in polynomial time.
2. For every LTL formula  $\varphi$ , there is an equivalent non-deterministic Büchi automaton<sup>4</sup>  $\mathcal{A}_\varphi$  of exponential size. States of  $\mathcal{A}_\varphi$  can be represented in polynomial space and given two states, one can check in polynomial time, whether one is a successor of the other.

These properties allow us to construct both  $t$  and a run of  $\mathcal{A}_\varphi$  on  $t$  on the fly. In detail, the algorithm works as follows. It guesses a set  $S^* \subseteq W$  and a state  $q^*$  of  $\mathcal{A}_\varphi$  and checks whether there are  $i < j$  satisfying the following properties:

- $S^* = S_i = S_j$ ,
- $q^*$  is reachable from the initial state of  $\mathcal{A}_\varphi$  by some run on the prefix  $t(0) \cdots t(i)$ , and
- $q^*$  is reachable from  $q^*$  by some run on the infix  $t(i+1) \cdots t(j)$ . This run has to visit at least one accepting state.

By an application of the pigeonhole principle, we can assume w.l.o.g. that  $j$  is at most exponential in  $|W|$  and in  $|\varphi|$ .

Let us argue that these properties can be checked in non-deterministic polynomial space. Given some guessed  $S^*$ , we can check the existence of  $i < j$  as required by computing the sequence  $S_0 S_1 S_2 \cdots$  on-the-fly, i.e., by just keeping the current set in memory, comparing it to  $S^*$ , then computing its successor, and then discarding the current set. While checking these reachability properties, the algorithm also guesses corresponding runs as required in the second and third property. As argued above, both tasks can be implemented in non-deterministic space. To ensure termination, we stop this search when the exponential upper bound on  $j$  is reached. This is possible using a counter with polynomially many bits and does not compromise completeness, as argued above.

It remains to argue that the algorithm is correct. First, assume  $t \stackrel{c}{\models} \varphi$ , which implies that  $\mathcal{A}_\varphi$  has an accepting run on  $t$ . Recall that  $t$  is ultimately periodic with characteristic  $(s, p)$  such that  $s + p \leq 2^{|W|}$  and that  $\mathcal{A}_\varphi$  is of exponential size. As a result, a pumping argument yields  $i < j$  with the desired properties.

Secondly, assume the algorithm finds  $i < j$  with the desired properties. Then, the run to  $q$  and the one from  $q$  to  $q$  can be turned into an accepting run of  $\mathcal{A}$  on  $t$ . That being so,  $t \stackrel{c}{\models} \varphi$ . ■

Note that our algorithm is even able to deal with arbitrary negations, as long as we disallow splitjunctions.

To conclude, let us quickly discuss the remaining obstacle to lift the algorithm to formulas with splitjunctions (but without arbitrary negations). As already alluded to above, one has to represent the result of splitting  $T(\mathcal{K})$  by two Kripke structures. Formally, the following property would be sufficient to obtain an algorithm for the full logic: if  $T(\mathcal{K}) \stackrel{s}{\models} \varphi_1 \vee \varphi_2$ , then there are Kripke structures  $\mathcal{K}_1$  and  $\mathcal{K}_2$  with  $T(\mathcal{K}_i) \stackrel{s}{\models} \varphi_i$  and  $T(\mathcal{K}) = T(\mathcal{K}_1) \cup T(\mathcal{K}_2)$ . Furthermore, as a crucial point, we have to require some upper bound on the size of the  $\mathcal{K}_i$ , which allows us to exhaustively search through the finite space of all structures of that size. In current research, we aim to prove this result.

## 5 LTL under Team Semantics vs. HyperLTL

LTL under team semantics expresses hyperproperties [6], i.e., sets of teams, or equivalently, sets of sets of traces. Recently, HyperLTL [5] was proposed to express information flow properties,

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<sup>4</sup>See, e.g., [1] for a formal definition of Büchi automata and for the construction of  $\mathcal{A}_\varphi$ .

which are naturally hyperproperties. For example, input determinism can be expressed as follows: every pair of traces that coincides on their input variables, also coincides on their output variables. To formalise such properties, HyperLTL allows to quantify over traces. This results in a powerful formalism with vastly different properties than LTL [15]. After introducing syntax and semantics of HyperLTL, we compare the expressive power of LTL under team semantics and HyperLTL.

The formulas of HyperLTL are given by the grammar

$$\begin{aligned}\varphi &::= \exists\pi. \varphi \mid \forall\pi. \varphi \mid \psi, \\ \psi &::= p_\pi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\psi \mid \psi\mathbf{U}\psi,\end{aligned}$$

where  $p$  ranges over atomic propositions in AP and where  $\pi$  ranges over a given countable set  $\mathcal{V}$  of *trace variables*. The other Boolean connectives and the temporal operators release R, eventually F, and always G are derived as usual, due to closure under negation. A sentence is a closed formula, i.e., one without free trace variables.

The semantics of HyperLTL is defined w.r.t. a trace assignment, a partial mapping  $\Pi: \mathcal{V} \rightarrow (2^{\text{AP}})^\omega$ . The assignment with empty domain is denoted by  $\Pi_\emptyset$ . Given a trace assignment  $\Pi$ , a trace variable  $\pi$ , and a trace  $t$ , denote by  $\Pi[\pi \rightarrow t]$  the assignment that coincides with  $\Pi$  everywhere but at  $\pi$ , which is mapped to  $t$ . Further,  $\Pi[i, \infty)$  denotes the assignment mapping every  $\pi$  in  $\Pi$ 's domain to  $\Pi(\pi)[i, \infty)$ . For teams  $T$  of traces and trace-assignments  $\Pi$  we define

$$\begin{aligned}(T, \Pi) &\models^h p_\pi && \text{if } p \in \Pi(\pi)(0), \\ (T, \Pi) &\models^h \neg\psi && \text{if } (T, \Pi) \not\models^h \psi, \\ (T, \Pi) &\models^h \psi_1 \vee \psi_2 && \text{if } (T, \Pi) \models^h \psi_1 \text{ or } (T, \Pi) \models^h \psi_2, \\ (T, \Pi) &\models^h \mathbf{X}\psi && \text{if } (T, \Pi[1, \infty)) \models^h \psi, \\ (T, \Pi) &\models^h \psi_1\mathbf{U}\psi_2 && \text{if there is a } k \geq 0 \text{ such that } (T, \Pi[k, \infty)) \models^h \psi_2 \text{ and} \\ &&& \text{for all } 0 \leq k' < k: (T, \Pi[k', \infty)) \models^h \psi_1, \\ (T, \Pi) &\models^h \exists\pi. \varphi && \text{if there is a trace } t \in T \text{ such that } (T, \Pi[\pi \rightarrow t]) \models^h \varphi, \text{ and} \\ (T, \Pi) &\models^h \forall\pi. \varphi && \text{if for all traces } t \in T: (T, \Pi[\pi \rightarrow t]) \models^h \varphi.\end{aligned}$$

We say that  $T$  satisfies a sentence  $\varphi$ , if  $(T, \Pi_\emptyset) \models^h \varphi$ , and write  $T \models^h \varphi$ . The semantics of HyperLTL are synchronous, i.e., the semantics of the until refers to a single  $k$ . Accordingly, one could expect that HyperLTL is closer related to LTL under synchronous team semantics than to LTL under asynchronous team semantics. In the following, we refute this intuition.

Formally, an HyperLTL sentence  $\varphi$  and an LTL formula  $\varphi'$  under synchronous (asynchronous) team semantics are equivalent, if for all teams  $T$ :  $T \models^h \varphi$  if and only if  $T \models^s \varphi'$  ( $T \models^a \varphi'$ ). In the following, let  $\forall$ -HyperLTL denote that set of HyperLTL sentences of the form  $\forall\pi. \psi$  with quantifier-free  $\psi$ , i.e., sentences with a single universal quantifier.

### Theorem 18

1. No LTL formula under synchronous or asynchronous team semantics is equivalent to  $\exists\pi. p_\pi$ .
2. No HyperLTL sentence is equivalent to  $\mathbf{F}p$  under synchronous team semantics.
3. LTL under asynchronous team semantics is as expressive as  $\forall$ -HyperLTL.

### PROOF

1. Consider  $T = \{\emptyset^\omega, \{p\}\emptyset^\omega\}$ . We have  $T \models^h \exists\pi. p_\pi$ . Assume there is an equivalent LTL formula under team semantics, call it  $\varphi$ . Then,  $T \models^s \varphi$  and thus  $\{\emptyset^\omega\} \models^s \varphi$  by downwards closure. Hence, by equivalence,  $\{\emptyset^\omega\} \models^h \exists\pi. p_\pi$ , yielding a contradiction.



2. Bozzelli et al. proved that the property encoded by  $Fp$  under synchronous team semantics cannot be expressed in HyperLTL [3].
3. Let  $\varphi$  be an LTL formula and define  $\varphi_h = \forall\pi. \varphi'$ , where  $\varphi'$  is obtained from  $\varphi$  by replacing each atomic proposition  $p$  by  $p_\pi$ . Then, due to singleton equivalence,  $T \models^a \varphi$  if and only if  $T \models^h \varphi_h$ . For the other implication, let  $\varphi = \forall\pi. \psi$  be a HyperLTL sentence with quantifier-free  $\psi$  and let  $\psi'$  be obtained from  $\psi$  by replacing each atomic proposition  $p_\pi$  by  $p$ . Then, again due to the singleton equivalence, we have  $T \models^h \varphi$  if and only if  $T \models^a \psi'$ . ■

Note that these separations are obtained by very simple formulas. In particular, the HyperLTL formulas are all negation-free.

**Corollary 19** *HyperLTL and LTL under synchronous team semantics are of incomparable expressiveness and HyperLTL is strictly more expressive than LTL under asynchronous team semantics.*

## 6 Conclusion

For all three considered decision problems the complexity of the asynchronous semantics coincides with the one of LTL formulas under classical semantics. Despite of that, the study of the complexities for the synchronous variant of the semantics gave compelling new insights into the interplay of formal language theoretic formalisms and team semantics. The path checking problem for this variant was shown to be PSPACE-complete, even when FO-definable atoms are allowed (Theorem 12); introducing arbitrary negation leads to PSPACE-completeness (Theorem 13). The model checking problem turned out to be more challenging to classify. We proved a PSPACE upper bound for the splitjunction-free fragment. This result can be extended to formulas using the splitjunction operator if there always exists a decomposition representable by Kripke structures of small size. Eventually, for LTL under team semantics as well as for HyperLTL there exist formulas which cannot be expressed in the other logic. Concluding, LTL under team semantics is a valuable logic which allows to express relevant hyperproperties and thereby complements the expressiveness of HyperLTL while allowing for computationally simpler decision problems.

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## References

- [1] Christel Baier and Joost-Pieter Katoen. *Principles of Model Checking*. The MIT Press, 2008.
- [2] Borzoo Bonakdarpour and Bernd Finkbeiner. Runtime verification for HyperLTL. In Yliès Falcone and César Sánchez, editors, *RV 2016*, volume 10012 of *LNCS*, pages 41–45. Springer, 2016.
- [3] Laura Bozzelli, Bastien Maubert, and Sophie Pinchinat. Unifying hyper and epistemic temporal logics. In Andrew M. Pitts, editor, *FoSSaCS 2015*, volume 9034 of *LNCS*, pages 167–182. Springer, 2015.
- [4] Julian Bradfield. On the structure of events in boolean games. In *Logics for Dependence and Independence*. Dagstuhl Reports, 2015.

- [5] Michael R. Clarkson, Bernd Finkbeiner, Masoud Koleini, Kristopher K. Micinski, Markus N. Rabe, and César Sánchez. Temporal logics for hyperproperties. In Martín Abadi and Steve Kremer, editors, *POST 2014*, volume 8414 of *LNCS*, pages 265–284. Springer, 2014.
- [6] Michael R. Clarkson and Fred B. Schneider. Hyperproperties. *Journal of Computer Security*, 18(6):1157–1210, 2010.
- [7] Pedro R. D’Argenio, Gilles Barthe, Sebastian Biewer, Bernd Finkbeiner, and Holger Hermanns. Is your software on dope? - formal analysis of surreptitiously ”enhanced” programs. In Hongseok Yang, editor, *ESOP 2017*, volume 10201 of *LNCS*, pages 83–110. Springer, 2017.
- [8] Stéphane Demri and Philippe Schnoebelen. The complexity of propositional linear temporal logics in simple cases. *Inf. Comput.*, 174(1):84–103, 2002.
- [9] Arnaud Durand, Miika Hannula, Juha Kontinen, Arne Meier, and Jonni Virtema. Approximation and dependence via multiteam semantics. In *FoIKS 2016*, pages 271–291, 2016.
- [10] Arnaud Durand, Juha Kontinen, and Heribert Vollmer. Expressivity and complexity of dependence logic. In *Dependence Logic: Theory and Applications*, pages 5–32. Birkhäuser, 2016.
- [11] Bernd Finkbeiner and Christopher Hahn. Deciding Hyperproperties. In Josée Desharnais and Radha Jagadeesan, editors, *CONCUR 2016*, volume 59 of *LIPICs*, pages 13:1–13:14, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.
- [12] Bernd Finkbeiner, Christopher Hahn, Marvin Stenger, and Leander Tentrup. Monitoring hyperproperties. In Shuvendu K. Lahiri and Giles Reger, editors, *RV 2017*, volume 10548 of *LNCS*, pages 190–207. Springer, 2017.
- [13] Bernd Finkbeiner, Christian Müller, Helmut Seidl, and Eugen Zalinescu. Verifying security policies in multi-agent workflows with loops. In *CCS 2017*. ACM, 2017. To appear.
- [14] Bernd Finkbeiner, Markus N. Rabe, and César Sánchez. Algorithms for model checking HyperLTL and HyperCTL\*. In Daniel Kroening and Corina S. Pasareanu, editors, *CAV 2015 (Part I)*, volume 9206 of *LNCS*, pages 30–48. Springer, 2015.
- [15] Bernd Finkbeiner and Martin Zimmermann. The first-order logic of hyperproperties. In Heribert Vollmer and Brigitte Vallée, editors, *STACS 2017*, volume 66 of *LIPICs*, pages 30:1–30:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.
- [16] Pietro Galliani. Inclusion and exclusion dependencies in team semantics - on some logics of imperfect information. *Ann. Pure Appl. Logic*, 163(1):68–84, 2012.
- [17] Erich Grädel, Juha Kontinen, Jouko Väänänen, and Heribert Vollmer. Logics for dependence and independence (Dagstuhl seminar 15261). *Dagstuhl Reports*, 5(6):70–85, 2015.
- [18] Erich Grädel and Jouko Väänänen. Dependence and independence. *Studia Logica*, 101(2):399–410, 2013.
- [19] Miika Hannula, Juha Kontinen, Jonni Virtema, and Heribert Vollmer. Complexity of propositional independence and inclusion logic. In Giuseppe F. Italiano, Giovanni Pighizzini, and Donald Sannella, editors, *MFCS 2015*, volume 9234 of *LNCS*, pages 269–280. Springer, 2015.

- [20] Miika Hannula, Juha Kontinen, Jonni Virtema, and Heribert Vollmer. Complexity of propositional logics in team semantics. *CoRR*, 1504.06135, 2015. Extended version of [19].
- [21] Wilfrid Hodges. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL*, 5(4):539–563, 1997.
- [22] Neil Immerman. *Descriptive Complexity*. Springer, 1998.
- [23] Andreas Krebs, Arne Meier, and Jonni Virtema. A team based variant of CTL. In Fabio Grandi, Martin Lange, and Alessio Lomuscio, editors, *TIME 2015*, pages 140–149. IEEE Computer Society, 2015.
- [24] Lars Kuhtz. *Model checking finite paths and trees*. PhD thesis, Saarland University, 2010.
- [25] Lars Kuhtz and Bernd Finkbeiner. LTL path checking is efficiently parallelizable. In Susanne Albers, Alberto Marchetti-Spaccamela, Yossi Matias, Sotiris E. Nikolettseas, and Wolfgang Thomas, editors, *ICALP 2009 (Part II)*, volume 5556 of *LNCS*, pages 235–246. Springer, 2009.
- [26] Antti Kuusisto. A Double Team Semantics for Generalized Quantifiers. *Journal of Logic, Language and Information*, 24(2):149–191, 2015.
- [27] Richard Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM Journal on Computing*, 6(3):467–480, 1977.
- [28] Nicolas Markey and Philippe Schnoebelen. Model checking a path. In Roberto M. Amadio and Denis Lugiez, editors, *CONCUR 2003*, volume 2761 of *LNCS*, pages 248–262. Springer, 2003.
- [29] Julian-Steffen Müller. *Satisfiability and Model Checking in Team Based Logics*. PhD thesis, Leibniz University of Hannover, 2014.
- [30] Amir Pnueli. The temporal logic of programs. In *FOCS 1977*, pages 46–57. IEEE Computer Society, 1977.
- [31] Markus N. Rabe. *A Temporal Logic Approach to Information-flow Control*. PhD thesis, Saarland University, 2016.
- [32] Ilya Shpitser. Causal inference and logics of dependence and independence. In *Logics for Dependence and Independence*. Dagstuhl Reports, 2015.
- [33] A. Prasad Sistla and Edmund M. Clarke. The complexity of propositional linear temporal logics. *J. ACM*, 32(3):733–749, 1985.
- [34] Jouko Väänänen. *Dependence Logic*. Cambridge University Press, 2007.
- [35] Jouko Väänänen. Modal Dependence Logic. In *New Perspectives on Games and Interaction*. Amsterdam University Press, Amsterdam, 2008.
- [36] Heribert Vollmer. *Introduction to Circuit Complexity – A Uniform Approach*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 1999.