# Visibly Linear Dynamic Logic*

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**Abstract.** We introduce Visibly Linear Dynamic Logic (VLDL), which is an extension of Linear Dynamic Logic (LDL) with temporal operators that are guarded by nondeterministic visibly pushdown automata. We prove that VLDL describes exactly the visibly pushdown languages over infinite words, which makes it strictly more powerful than LTL and LDL and able to express properties of recursive programs. The main technical contribution of this work is a translation of VLDL formulas into nondeterministic visibly pushdown automata with Büchi acceptance of exponential size via one-way alternating jumping automata.
This translation yields exponential-time algorithms for satisfiability, validity and model checking. We also show that visibly pushdown games with VLDL winning conditions are solvable in triply-exponential time. We show all of these problems to be complete for their respective complexity classes. Furthermore, we prove that using deterministic pushdown automata as guards yields undecidable decision problems.

## 1 Introduction

Linear Temporal Logic (LTL) [14] is widely used for the specification of system behavior. Its popularity is mainly due to its simple syntax and intuitive semantics, as well as its exponential compilation property, i.e., for each LTL formula there exists an equivalent Büchi automaton of exponential size. Due to the latter property, there exist algorithms for model checking of properties specified in LTL in polynomial space and for checking realizability in doubly exponential time.

LTL, however, has the significant weakness of not even being powerful enough to express all $\omega$-regular properties. There are several approaches to address this shortcoming, by augmenting LTL, for example, with regular expressions [11], finite automata on infinite words [16], and right-linear grammars [17]. We concentrate on the approach of Linear Dynamic Logic (LDL) [15], which extends the globally- and eventually-operators of LTL with regular expressions with tests. While the LTL-formula $\mathbf{F}\psi$ simply means "either now, or at some point in the future, $\psi$ holds", the corresponding LDL operator $\langle r \rangle \psi$ means "There exists an infix starting at the current position, which matches $r$, and $\psi$ holds after that prefix".

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The logic LDL captures the $\omega$-regular languages. In spite of its greater expressive power, LDL still enjoys the exponential compilation property, whence there exist algorithms for model checking and realizability in polynomial space and doubly-exponential time, respectively. While the expressive power of LDL is sufficient for many specifications, it is not possible to reason about recursive properties of systems, such as statements concerning the call stack. In this work, we address this shortcoming by combining the temporal operators of LDL with visibly pushdown automata to obtain visibly Linear Dynamic Logic (VLDL).

A visibly pushdown automaton (VPA) [1] is a pushdown automaton that has a fixed partition of the input alphabet into calls, returns and local actions. In contrast to traditional pushdown automata, VPAs may only push symbols onto the stack when reading calls and may only pop symbols off the stack when reading returns. Moreover, they may not inspect the topmost symbol of the stack when not reading returns. Thus, the height of the stack after reading a word is known in advance for all VPAs using the same partition of the input alphabet. Due to this, VPAs are closed under union and intersection, as well as complementation. These automata are known to allow for specification of many properties of interest in program verification such as "there are infinitely many positions at which at most two functions are active" or "every time the program enters a module $m$, if $p$ holds true, then $p$ holds true upon exiting $m$" [1]. By combining the temporal operators of LDL with VPAs, VLDL allows for modular specification of such properties while still retaining the expressive power of VPAs.

**Our contributions** Firstly, we describe translations from VPAs to VLDL and vice versa. For the direction from automata to logic we use a translation of VPAs into deterministic parity stair automata (PSA) by Löding et al. [12], which we then translate into VLDL formulas. For the direction from logic to automata we translate VLDL formulas into one-way alternating jumping automata, which are known to be translatable into VPAs of exponential size due to Bozzelli [4].

Secondly, we show that the satisfiability problem and the validity problem are ExpTime-complete. Membership in ExpTime follows from the previously mentioned constructions, while we show ExpTime-hardness of the problems by adapting a proof of ExpTime-hardness of model checking pushdown systems against LTL specifications by Bouajjani et al. [3].

Thirdly, we show that model checking visibly pushdown systems against VLDL specifications is ExpTime-complete as well, where membership in ExpTime and ExpTime-hardness follow from ExpTime-membership of the emptiness problem and ExpTime-hardness of the validity problem for VLDL.

As a fourth result, we show that solving visibly pushdown games against VLDL winning conditions is 3ExpTime-complete. Membership in 3ExpTime follows from the aforementioned translation of VLDL formulas into VPAs and the membership of solving pushdown games against VPA winning conditions in 2ExpTime due to Löding et al. [12]. 3ExpTime-hardness of the problem is

due to a reduction from the problem of solving pushdown games against LTL specifications, again due to Löding et al [12].

Finally, we show that replacing the visibly pushdown automata used as guards in VLDL by deterministic pushdown automata yields a logic with an undecidable emptiness problem.

These results show that VLDL allows for the concise specification of important properties in a logic with intuitive semantics. In the case of the satisfiability problem, we move from PSPACE-completeness to EXPTIME-completeness by replacing regular expressions with VPAs. For the realizability problem, we gain an exponent and move from 2EXPTIME-completeness of the problem for LDL to 3EXPTIME-completeness of the problem for VLDL. Moreover, strengthening VLDL by using traditional pushdown automata results in a logic with an undecidable satisfiability problem, even if we restrict ourselves to deterministic pushdown automata.

**Related Work** There exist other approaches to using visibly pushdown languages for specifications, most notably CaRet [2], and, more recently, VLTL [5]. While VLTL captures the class of visibly pushdown languages, CaRet captures only a strict subset of it. For both logics there exist exponential translations to VPAs. In this work, we provide exponential translations from VLDL to VPAs and vice versa. Hence, CaRet is strictly less powerful than VLDL, but every CaRet formula can be translated into an equivalent VLDL formula with a doubly-exponential blowup. Similarly, every VLTL formula can be translated into an equivalent VLDL formula and vice versa, both with doubly-exponential blowup.

Other logical characterizations of visibly pushdown languages include characterizations by monadic second order logic augmented with a binary matching predicate ($\text{MSO}_\mu$)[1] and by a fixpoint logic [4]. Even though these logics also capture the class of visibly pushdown languages, they feature neither an intuitive syntax nor semantics and thus are less usable than VLDL.

Temporal operators are combined with visibly pushdown automata in an extension of the branching-time logic PDL [13]. Due to the different setting, however, this logic is incomparable to VLDL. In contrast to the results for VLDL, the decision problems for the logic from [13] are 2EXPTIME-complete.

## 2 Preliminaries

In this section we introduce the basic notions used in the remainder of this work, namely (nondeterministic) visibly pushdown automata and related concepts. A pushdown alphabet $\widetilde{\Sigma} = (\Sigma_c, \Sigma_r, \Sigma_l)$ is a finite set $\Sigma$ that is partitioned into calls $\Sigma_c$, returns $\Sigma_r$ and local actions $\Sigma_l$. We write $w = w_0 \cdots w_n$ and $\alpha = \alpha_0\alpha_1\alpha_2 \cdots$ to denote finite and infinite words, respectively. Additionally, we denote the stack height reached by any automaton after reading $w$ by $sh(w)$ which is defined inductively as $sh(\varepsilon) = 0$, $sh(wc) = sh(w) + 1$ for $c \in \Sigma_c$, $sh(wr) = \max\{0, sh(w) - 1\}$ for $r \in \Sigma_r$, and $sh(wl) = sh(w)$ for $l \in \Sigma_l$. We

say that a call $c$ at some position $k$ of a word $w$ is matched if there exists an $l > k$ with $w_l \in \Sigma_r$ and $sh(w_0 \cdots w_k) = sh(w_0 \cdots w_l)$ and call the return at the smallest such position $l$ the matching return of $c$. Moreover, we define $steps(\alpha) := \{k \in \mathbb{N} \mid \forall l \geq k.\ sh(\alpha_0 \cdots \alpha_l) \geq sh(\alpha_0 \cdots \alpha_k)\}$ as those positions that reach a new lower bound for the stack height. Note that $0 \in steps(\alpha)$ and that $steps(\alpha)$ is infinite for all infinite words $\alpha$. For finite words $w$, $steps(w)$ is finite and $|w| \in steps(w)$.

**Visibly Pushdown Systems** We consider several models of automata, all of which have visibly pushdown systems as their underlying technical core. A visibly pushdown system (VPS) $\mathcal{S} = (Q, \widetilde{\Sigma}, \Gamma, \Delta)$ consists of a finite set of states $Q$, a pushdown alphabet $\widetilde{\Sigma}$, a stack alphabet $\Gamma$, which contains a stack-bottom marker $\bot$, and a transition relation

$$\Delta \subseteq (Q \times \Sigma_c \times Q \times (\Gamma \setminus \{\bot\})) \cup (Q \times \Sigma_r \times \Gamma \times Q) \cup (Q \times \Sigma_l \times Q).$$

A configuration $(q, \gamma)$ of $\mathcal{S}$ is a pair of a state $q \in Q$ and a stack content $\gamma \in (\Gamma \setminus \{\bot\})^* \cdot \bot$. The VPS $\mathcal{S}$ induces the configuration graph $G_{\mathcal{S}} = (Q \times \Gamma^*, E)$ with $E \subseteq ((Q \times \Gamma^*) \times \Sigma \times (Q \times \Gamma^*))$ and $((q, \gamma), a, (q', \gamma')) \in E$ if and only if either

- $a \in \Sigma_c$, $(q, a, q', A) \in \Delta$, and $A\gamma = \gamma'$,
- $a \in \Sigma_r$, $(q, a, \bot, q') \in \Delta$, and $\gamma = \gamma' = \bot$,
- $a \in \Sigma_r$, $(q, a, A, q') \in \Delta$, $A \neq \bot$, and $\gamma = A\gamma'$, or
- $a \in \Sigma_l$, $(q, a, q') \in \Delta$, and $\gamma = \gamma'$.

For an edge $e = ((q, \gamma), a, (q', \gamma'))$ we say that $a$ is the label of $e$. A run $\pi = (q_0, \gamma_0) \cdots (q_n, \gamma_n)$ of $\mathcal{S}$ on the word $w = w_0 \cdots w_{n-1}$ is a sequence of configurations such that for all $i \in [0; n-1]$ there is an $\alpha_i$ labeled edge from $(q_i, \gamma_i)$ to $(q_{i+1}, \gamma_{i+1})$ in $G_{\mathcal{S}}$. We say that $\mathcal{S}$ is deterministic if for each state $q$, each stack content $\gamma$ and each symbol $a \in \Sigma$ there exists at most one outgoing $a$-labeled edge from $(q, \gamma)$ in $G_{\mathcal{S}}$. When drawing VPS's, we write $\downarrow A$, $\uparrow A$ and $\rightarrow$ to denote pushing $A$ onto the stack, popping $A$ from the stack, and local actions, respectively.

**(Büchi) Visibly Pushdown Automata** A (nondeterministic) visibly pushdown automaton (VPA) [1] is a six-tuple $\mathfrak{A} = (Q, \widetilde{\Sigma}, \Gamma, \Delta, Q_I, F)$, where the four-tuple $(Q, \widetilde{\Sigma}, \Gamma, \Delta)$ is a VPS and $Q_I, F \subseteq Q$ are sets of initial and final states. A run of $\mathfrak{A}$ is called initial if $(q_0, \gamma_0) = (q_I, \bot)$ for some $q_I \in Q_I$. A finite run $\pi = (q_0, \gamma_0) \cdots (q_n, \gamma_n)$ is accepting if $q_n \in F$. A Büchi VPA (BVPA) is syntactically identical to a VPA, but we only consider runs over infinite words. An infinite run is Büchi-accepting if it visits states in $F$ infinitely often. A (B)VPA $\mathfrak{A}$ accepts a word $w(\alpha)$ if there exists an initial (Büchi-)accepting run of $\mathfrak{A}$ on $w(\alpha)$. The family of languages accepted by (B)VPA is called ($\omega$-)VPL. If it is clear from the context, we omit the prefix "Büchi" and call both models VPA.

## 3 Visibly Linear Dynamic Logic

We fix a finite set $P$ of atomic propositions and a partition $\widetilde{\Sigma} = (\Sigma_c, \Sigma_r, \Sigma_l)$ of $2^P$ throughout the remainder of this work. The syntax of VLDL is defined by the grammar

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle\mathfrak{A}\rangle\varphi \mid [\mathfrak{A}]\varphi,$$

where $p \in P$ and $\mathfrak{A}$ ranges over testing (nondeterministic) visibly pushdown automata (TVPA) over the fixed alphabet $\widetilde{\Sigma}$. A TVPA $\mathfrak{A} = (Q, \widetilde{\Sigma}, \Gamma, \Delta, Q_I, F, t)$ consists of a VPA $(Q, \widetilde{\Sigma}, \Gamma, \Delta, Q_I, F)$ and a partial function $t$ mapping states to VLDL formulas over $\widetilde{\Sigma}$. [1] Intuitively, such an automaton accepts a substring $\alpha_i \cdots \alpha_j$ of an infinite word $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ if the embedded VPA accepts $\alpha_i \dots \alpha_j$ and, if the state $q_k$ of the accepting run $(q_0, \gamma_0)\cdots(q_{j-i+1}, \gamma_{j-i+1})$ is marked with the test $\varphi$ by $t$, then the suffix $\alpha_{i+k}\alpha_{i+k+1}\alpha_{i+k+2}\cdots$ of $\alpha$ satisfies $\varphi$.

We write $\mathrm{cl}(\varphi)$ to denote all subformulas of $\varphi$, including those contained as tests in automata and their subformulas, and define the size of $\varphi$ as the sum of $|\mathrm{cl}(\varphi)|$ and the sum of the numbers of states of the automata contained in $\varphi$. As shorthands, we use $\mathtt{tt} := p \vee \neg p$ and $\mathtt{ff} := p \wedge \neg p$ for some atomic proposition $p$. Even though the testing function $t$ is defined as a partial function, we generally assume it is total by setting $t\colon q \mapsto \mathtt{tt}$ if $q \notin \mathit{domain}(t)$.

Let $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ be an infinite word over $2^P$ and let $k \in \mathbb{N}$ be a position in $\alpha$. We define the semantics of VLDL inductively over the structure of a formula $\varphi$ via

- $(\alpha, k) \models p$ if $p \in \alpha_k$,
- $(\alpha, k) \models \neg\varphi$ if $(\alpha, k) \not\models \varphi$,
- $(\alpha, k) \models \varphi_0 \wedge \varphi_1$ if $(\alpha, k) \models \varphi_0$ and $(\alpha, k) \models \varphi_1$, and dually for $\varphi_0 \vee \varphi_1$,
- $(\alpha, k) \models \langle\mathfrak{A}\rangle\varphi$ if there exists $l \geq k$ s.t. $(k, l) \in \mathcal{R}_{\mathfrak{A}}(\alpha)$ and $(\alpha, l) \models \varphi$,
- $(\alpha, k) \models [\mathfrak{A}]\varphi$ if for all $l \geq k$, $(k, l) \in \mathcal{R}_{\mathfrak{A}}(\alpha)$ implies $(\alpha, l) \models \varphi$,

where the relation $\mathcal{R}_{\mathfrak{A}}(\alpha)$ contains all pairs $(k, l)$ such that the TVPA $\mathfrak{A}$ accepts $\alpha_k \dots \alpha_{l-1}$. Formally, $\mathcal{R}_{\mathfrak{A}}(\alpha)$ is defined as

$$\begin{aligned}\mathcal{R}_{\mathfrak{A}}(\alpha) := \{(k, l) \in \mathbb{N} \times \mathbb{N} \mid\ & \exists \text{ acc. run } (q_0, \sigma_0)\cdots(q_{l-k}, \sigma_{l-k}) \text{ of } \mathfrak{A} \text{ on } \alpha_k \cdots \alpha_{l-1} \\ & \textbf{and } \forall m \in [0; l-k].\ (\alpha, l+m) \models t(q_m)\}.\end{aligned}$$

We write $\alpha \models \varphi$ as a shorthand for $(\alpha, 0) \models \varphi$ and say that $\alpha$ is a model of $\varphi$ in this case. The language of $\varphi$ is defined as $L(\varphi) := \{\alpha \in (2^P)^\omega \mid \alpha \models \varphi\}$. As usual, disjunction and conjunction are dual, as well as the $\langle\mathfrak{A}\rangle$-operator and the $[\mathfrak{A}]$-operator, which can be dualized using De Morgan's law and the logical identity $[\mathfrak{A}]\varphi \equiv \neg\langle\mathfrak{A}\rangle\neg\varphi$. Note that the latter identity only dualizes the temporal

[1] Obviously, there are some restrictions on the nesting of tests into automata, e.g., $\varphi = \langle\mathfrak{A}\rangle\varphi'$ may not appear as a test in $\mathfrak{A}$.

operator, but does not require complementation of the automaton guarding the operator. We additionally allow the use of derived boolean operators such as $\rightarrow$ and $\leftrightarrow$, as they can easily be reduced to the basic operators $\wedge$, $\vee$ and $\neg$.

The logic VLDL combines the expressive power of visibly pushdown automata with the intuitive temporal operators of LDL. Thus it allows for concise and intuitive specifications of many interesting properties in program verification [1]. In particular, VLDL allows for the specification of properties of recursive programs, which makes it more expressive than both LDL and LTL. In fact, we can embed LDL in VLDL in linear time. We show strictness of this inclusion in Section 5.

**Lemma 1.** *For any LDL formula $\psi$ over $P$ we can effectively construct a VLDL formula $\varphi$ over $\widetilde{\Sigma} := (\emptyset, \emptyset, 2^P)$ in linear time such that $L(\psi) = L(\varphi)$.*

*Proof.* Let $\psi$ be an LDL formula over the alphabet $P$. The only interesting case is $\psi = \langle r \rangle \psi'$, since all other cases follow from closure properties and duality. We obtain the VLDL formula $\varphi'$ over $\widetilde{\Sigma}$ equivalent to $\psi'$ by induction and construct the finite automaton $\mathfrak{A}_r$ from $r$ using the construction from [9]. The automaton $\mathfrak{A}_r$ contains tests, but is not equipped with a stack. Since $\widetilde{\Sigma} = (\emptyset, \emptyset, 2^P)$, we can interpret $\mathfrak{A}_r$ as a TVPA without changing the language is recognizes. We call the TVPA $\mathfrak{A}'_r$ and define $\varphi = \langle \mathfrak{A}'_r \rangle \varphi'$. □

Since LTL can be in turn embedded in LDL, Lemma 1 directly implies the embeddability of LTL in VLDL.

**Proposition 1.** *For any LTL formula $\psi$ over $P$ we can effectively construct a VLDL formula $\varphi$ over $\widetilde{\Sigma} := (\emptyset, \emptyset, 2^P)$ in linear time such that $L(\psi) = L(\varphi)$.*

## 4 Examples

As we show in Section 5, VLDL captures the visibly pushdown languages and thus, it is strictly more expressive than traditional Büchi automata. In fact, VLDL allows for concise formulations of a number of important properties of recursive programs in program verification. We give some examples of such properties and their formalization in this section.

*Example 1.* Assume that we have a program that uses some module $m$ and has the observable atomic propositions $P := \{\texttt{call}_m, \texttt{return}_m, p\}$, where $\texttt{call}_m$ and $\texttt{return}_m$ denote calls to and returns from $m$, while $p$ is some arbitrary atomic proposition.

We now construct a formula that describes the condition "If $p$ holds true immediately after entering $m$, it shall hold as well immediately after the corresponding return from $m$" [2]. To this end, we pick the pushdown alphabet $\widetilde{\Sigma} = (\Sigma_c, \Sigma_r, \Sigma_l) := (\{\texttt{call}_m\}, \{\texttt{return}_m\}, \{p\})$. For the sake of readability we ignore the other members of $2^P$. The formula $\varphi := [\mathfrak{A}_c](p \rightarrow \langle \mathfrak{A}_r \rangle p)$ then captures the condition, with $\mathfrak{A}_c$ and $\mathfrak{A}_r$ as shown in Figure 1. The automaton $\mathfrak{A}_c$ accepts

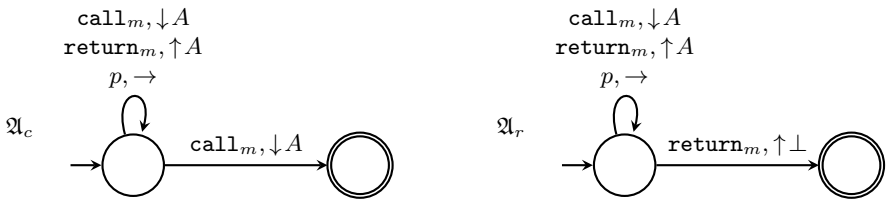


**Fig. 1.** The automata $\mathfrak{A}_c$ and $\mathfrak{A}_r$ for Example 1.

all finite words ending with a call to $m$, whereas the automaton $\mathfrak{A}_r$ accepts all words ending with a single unmatched return.

The formula $\varphi$ has the subformulas $\mathrm{cl}(\varphi) = \{\varphi, \neg p \vee \langle \mathfrak{A}_r \rangle p, \neg p, p\}$ and is of size $|\varphi| = |\mathrm{cl}(\varphi)| + |\mathfrak{A}_c| + |\mathfrak{A}_r| = 3 + 2 + 2 = 7$.

Note that the stack is simply used as a counter in the previous example. This technique suffices for the specification of other properties as well, such as tracking the path through the directory structure instead of the call stack.

*Example 2.* We consider a simplified system model, in which a user can move through directories and obtain and relinquish superuser rights. To this end, we consider the set of atomic propositions $P = \{\mathtt{cd}_\downarrow, \mathtt{cd}_\uparrow, \mathtt{sudo}, \mathtt{logout}\}$, where $\mathtt{cd}_\downarrow$ denotes moving into a subdirectory of the current working directory, $\mathtt{cd}_\uparrow$ denotes moving to the parent directory, `sudo` denotes the acquisition of elevated privileges, and `logout` denotes relinquishing them. We again only define the pushdown alphabet for singleton subsets of $P$ for readability and pick $\widetilde{\Sigma} := (\{\mathtt{cd}_\downarrow\}, \{\mathtt{cd}_\uparrow\}, \{\mathtt{sudo}, \mathtt{logout}\})$ in order to formalize the property "If the program acquires elevated privileges, it has to relinquish them before moving out of its current directory" [8].

We use the stack as a counter using the stack alphabet $\Gamma := \{\bot, A\}$. Then the formula $\varphi := [\mathfrak{A}_{priv}] \neg (\langle \mathfrak{A}_{par} \rangle \mathtt{tt})$, specifies the property above, where $\mathfrak{A}_{priv}$ accepts all prefixes of runs of the program that end with the acquisition of elevated privileges, and $\mathfrak{A}_{par}$ tracks the depth of the current working directory. Figure 2 depicts the automata $\mathfrak{A}_{priv}$ and $\mathfrak{A}_{par}$.

While the previous example shows how to handle programs that can simply request a single set of elevated rights, in actual systems the situation is more intricate. In reality, a program may request the rights of any user of the system by logging in as that user. When logging out, the rights revert to those of the previously logged in user. In the following example we use the stack to keep track of the currently logged in user and ensure that system calls are not executed with elevated privileges.

*Example 3.* In this example we remove some of the simplifications of the previous example and model the login mechanism of an actual system more precisely. To this end, let $P = \{\mathtt{exec}, \mathtt{login}_s, \mathtt{login}_u, \mathtt{logout}\}$, where `exec` denotes the execution of a system call, $\mathtt{login}_s$ and $\mathtt{login}_u$ denote the login as

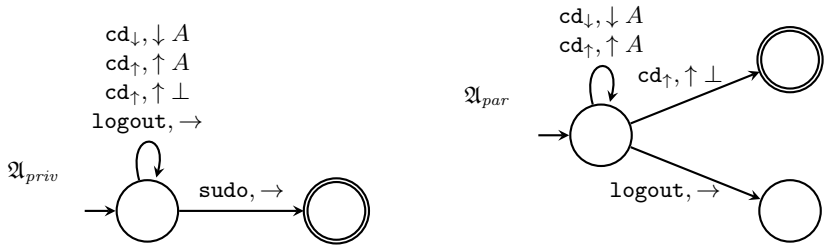


**Fig. 2.** The automata $\mathfrak{A}_{priv}$ and $\mathfrak{A}_{par}$ for Example 2.

the superuser and some other user, respectively, and `logout` denotes logging the current user out and reverting to the previous user. The pushdown alphabet $\widetilde{\Sigma} := (\{\mathtt{login}_s, \mathtt{login}_u\}, \{\mathtt{logout}\}, \{\mathtt{exec}\})$ allows us to keep track of the stack of logged in users. We want to specify the property "Whenever the program has obtained elevated privileges, it does not leave the directory it originally obtained these privileges in."

Recall that visibly pushdown automata are not allowed to inspect the top of the stack. Thus, in order to correctly trace the currently logged in user, we need to store both the current user and the previously logged in one on the stack. The automaton $\mathfrak{A}_{user}$ performs this bookkeeping using the stack alphabet $\Gamma := \{(c,p) \mid c,p \in \{s,u\}\}$, where $c$ denotes the currently logged in user, and $p$ denotes the previously logged in user. It moves to the state $u$ when a normal user is logged in and to state $s$ when a superuser is logged in.

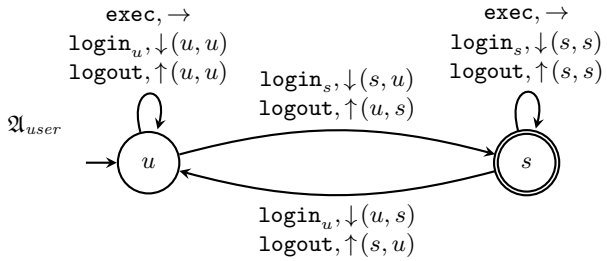


**Fig. 3.** The automaton $\mathfrak{A}_{user}$, which keeps track of the status of the currently logged in user.

Since the only action available to the program in this example apart from logging users in or out is to execute system calls, we do not need an automaton to capture the undesired behavior, but can simply use the atomic proposition `exec` in the formula. Hence, the formula $\varphi := [\mathfrak{A}_{user}] \neg \mathtt{exec}$ defines the desired behavior.

Due to the modular nature of VLDL, we can easily reuse existing automata and subformulas. Consider, for example, that we want to make sure that superusers neither execute system calls, nor leave the directory they were in when they acquired superuser-privileges. Using some simple modifications to $\mathfrak{A}_{user}$ and $\mathfrak{A}_{par}$ to work over an extended set of atomic propositions, we can specify the conjunction of the previously defined behaviors without having to construct new automata from scratch.

## 5 VLDL Captures $\omega$-VPL

In this section we show that VLDL is as expressive as $\omega$-VPL. Recall that a language $L$ is in $\omega$-VDL if and only if there exists a BVPA recognizing $L$. We provide effective constructions for transforming BVPAs into equivalent VLDL formulas and vice versa. This shows that VLDL captures $\omega$-VPL.

**Theorem 1.** *For any language of infinite words $L \subseteq \Sigma^\omega$ there exists a BVPA $\mathfrak{A}$ with $L(\mathfrak{A}) = L$ if and only if there exists a VLDL formula $\varphi$ with $L(\varphi) = L$. There exist effective translations for both directions.*

In Section 5.1 we show the construction of VLDL formulas from BVPAs via deterministic parity stair automata. In Section 5.2 we construct one-way alternating jumping automata from VLDL formulas, which are known to be translatable into equivalent BVPAs. Both constructions incur an exponential blowup in size. In the construction of BVPAs from VLDL formulas, this blowup is shown to be unavoidable.

### 5.1 From Stair Automata to VLDL

In this section we construct a VLDL formula of exponential size that is equivalent to a given BVPA $\mathfrak{A}$. To this end, we first transform $\mathfrak{A}$ into an equivalent deterministic parity stair automaton (DPSA) [12] $\mathfrak{A}_{st}$. A parity stair automaton $\mathfrak{A} = (Q, \widetilde{\Sigma}, \Gamma, \Delta, Q_I, \Omega)$ consists of

- a VPS $\mathcal{S} = (Q, \widetilde{\Sigma}, \Gamma, \Delta)$,
- a set of initial states $Q_I$,
- and a coloring $\Omega \colon Q \to \mathbb{N}$.

It is called deterministic if $\mathcal{S}$ is deterministic and we have $|Q_I| = 1$.

A run $\rho_\alpha$ of $\mathfrak{A}$ on an infinite word $\alpha$ is a run of the VPS $\mathcal{S}$ on $\alpha$. The run $\rho_\alpha = (q_0, \sigma_0)(q_1, \sigma_1)(q_2, \sigma_2)\cdots$ on the word $\alpha$ induces a sequence of colors $\Omega(\rho_\alpha) := \Omega(q_{k_0})\Omega(q_{k_1})\Omega(q_{k_2})\cdots$, where $k_0 < k_1 < k_2 \cdots$ is an ordered enumeration of the steps of $\alpha$. A stair automaton $\mathfrak{A}$ accepts an infinite word $\alpha$ if there exists an initial run $\rho$ of $\mathfrak{A}$ on $\alpha$ such that the largest color appearing infinitely often in $\Omega(\rho)$ is even. The language $L(\mathfrak{A})$ of a parity stair automaton $\mathfrak{A}$ is the set of all words $\alpha$ that are accepted by $\mathfrak{A}$.

**Lemma 2 ([12]).** *For every BVPA $\mathfrak{A}$ there exists an effectively constructible equivalent deterministic parity stair automaton $\mathfrak{A}_{st}$. The size of $\mathfrak{A}_{st}$ is exponential in the size of $\mathfrak{A}$.*

Since the stair automaton $\mathfrak{A}_{st}$ equivalent to a BVPA $\mathfrak{A}$ is deterministic, the acceptance condition collapses to the requirement that the unique run of $\mathfrak{A}_{st}$ on $\alpha$ must be accepting. Another important observation is that every time $\mathfrak{A}_{st}$ reaches a step of $\alpha$, it is possible to clear the stack. Since the topmost element of the stack will never be popped after reaching a step, and since VPAs cannot inspect the top of the stack, neither this symbol, nor the ones below it have any influence on the remainder of the run. We use these two observations to characterize acceptance by $\mathfrak{A}_{st}$ by a VLDL formula.

**Lemma 3.** *For each DPSA $\mathfrak{A}_{st}$ there exists an effectively constructible equivalent VLDL formula $\varphi_{\mathfrak{A}_{st}}$. The size of $\varphi_{\mathfrak{A}_{st}}$ is quadratic in the number of states of $\mathfrak{A}$.*

*Proof.* We first construct a formula $\varphi^t_{st}$ such that $(\alpha, k) \models \varphi^t_{st}$ if and only if $k \in \mathit{steps}(\alpha)$. Let $\mathfrak{A}^t_{st}$ be a VPA that accepts upon reading an unmatched return, constructed similarly to $\mathfrak{A}_r$ from Example 1. We define $\varphi^t_{st} := [\mathfrak{A}^t_{st}]\mathtt{ff}$, since we demand that the stack height never drops below the current level, i.e., $\mathfrak{A}_{st}$ never accepts. Then $\varphi^t_{st}$ has the desired property.

In the remainder of this proof, we write ${}_q\mathfrak{A}_{q'}$ to denote the TVPA that we obtain from combining the VPS $\mathcal{S}$ of $\mathfrak{A}$ with the sets of initial and final states $\{q\}$ and $\{q'\}$, as well as the testing function $t$ that marks all states with the test $\mathtt{tt}$. Additionally we require that ${}_q\mathfrak{A}_{q'}$ does not accept the empty word. This is trivial if $q$ and $q'$ are different, and easily achieved by adding a new initial state if $q = q'$. Furthermore, we define $Q_{even} := \{q \in Q \mid \Omega(q) \text{ is even}\}$ and $Q_{>c} := \{q \in Q \mid \Omega(q) > c\}$. For any automaton $\mathfrak{A}$ we write $\mathfrak{A}'$ to denote the automaton that we obtain by adding copies of the final states of $\mathfrak{A}$, which are marked with the test $\varphi^t_{st}$, inherit all incoming transitions of the original final states, but have no outgoing transitions. The former construction has precedence over the latter, so ${}_q\mathfrak{A}'_{q'}$ should be read as $({}_q\mathfrak{A}_{q'})'$.

Recall that $\mathfrak{A}$ accepts a word $\alpha$ if the largest color seen infinitely often at a step during the unique run of $\mathfrak{A}$ on $w$ is even. This is equivalent to the condition that there exists a state $q$ with even color, such that after some finite prefix which ends at a step in $q$, no state with larger color than $\Omega(q)$ is visited at a step, and that every visit to $q$ at a step is followed by another visit to $q$ at a step. The former condition is formalized as $\varphi_1(q) := \langle {}_{q_I}\mathfrak{A}'_q\rangle(\bigwedge_{q' \in Q_{>\Omega(q)}} [{}_q\mathfrak{A}'_{q'}]\mathtt{ff})$, the latter one as $\varphi_2(q) := [\mathfrak{A}'_q]\langle {}_q\mathfrak{A}'_q\rangle\mathtt{tt}$. Acceptance of $\mathfrak{A}$ is then captured by $\varphi_{\mathfrak{A}} := \bigvee_{q \in Q_{even}} \varphi_1(q) \wedge \varphi_2(q)$.

The construction of $\varphi_2(q)$ relies heavily on the determinism of the DPSA $\mathfrak{A}$. If $\mathfrak{A}$ were not deterministic, then the universal quantification over all runs ending in $q$ at a step would also capture partial runs that are eventually rejecting. Since we only have a single run of $\mathfrak{A}$ on the input word, however, this construction yields a formula with the intended meaning. Furthermore, both $\varphi_1(q)$ and $\varphi_2(q)$

use the observation that we are able to clear the stack every time that we reach a step. Thus, even though the stack contents of ${}_{q_I}\mathfrak{A}'_q$ and $\mathfrak{A}'_q$ are not carried over to ${}_q\mathfrak{A}'_q$ and ${}_q\mathfrak{A}'_{q'}$, the concatenation of the automata does not change the resulting run.

We have $\alpha \in L(\mathfrak{A})$ if and only if $(w, 0) \models \varphi_{\mathfrak{A}}$ and hence $L(\mathfrak{A}) = L(\varphi_{\mathfrak{A}})$. Thus, the claim holds true. □

By combining Lemmas 2 and 3 we see that VLDL is at least as expressive as VPA. The construction inherits an exponential blowup from the construction of DPSAs from BVPAs. This proves the direction from automata to logic of Theorem 1.

In the next section we show that each VLDL formula $\varphi$ can be transformed into an equivalent VPA. Thus, the construction from the proof of Lemma 3 yields a normal form for VLDL formulas. Since the tests of the automata contained in this formula only occur in final states, we can lift them from the automaton to the formula, i.e., $\langle\mathfrak{A}'\rangle\varphi$ and $[\mathfrak{A}']\varphi$ become $\langle\mathfrak{A}\rangle(\varphi^t_{st} \wedge \varphi)$ and $[\mathfrak{A}](\varphi^t_{st} \to \varphi)$, respectively. Moreover, we can reduce $\bigwedge_{q' \in Q_{>\Omega(q)}} [{}_q\mathfrak{A}'_{q'}]\mathtt{ff}$ to $[{}_q\mathfrak{A}'_{Q_{>\Omega(q)}}]\mathtt{ff}$, by adapting the notation for replacing final states to sets of final states in the obvious way.

**Proposition 2.** *Let $\varphi$ be a VLDL formula. There exists an equivalent formula*

$$\varphi' = \bigvee_{i=1}^{n} \langle\mathfrak{A}^i_1\rangle(\varphi_{st} \wedge [\mathfrak{A}^i_2]\mathit{ff}) \wedge [\mathfrak{A}^i_3](\varphi_{st} \to \langle\mathfrak{A}^i_4\rangle\varphi_{st}),$$

*for some $n \in \mathbb{N}$, where all $\mathfrak{A}^i_j$ share the same underlying VPS, none of the $\mathfrak{A}^i_j$ contain tests, and $\varphi_{st}$ is fixed over all $\varphi$.*

### 5.2 From VLDL to 1-AJA

We now consider the construction of an equivalent BVPA $\mathfrak{A}_\varphi$ from a VLDL formula $\varphi$. To this end, we first construct a one-way alternating jumping automaton (1-AJA) [4] $\mathfrak{A}_{aja}$ from $\varphi$, which is known to be translatable into an equivalent BVPA in exponential time. For the remainder of this section, we define $\mathit{Comms}_Q := \{\to, \to_a\} \times Q \times Q$.

A 1-AJA $\mathfrak{A} = (Q, \widetilde{\Sigma}, Q_I, \delta, \Omega)$ consists of

- a finite state set $Q$,
- a visibly pushdown alphabet $\widetilde{\Sigma}$,
- a set $Q_I \subseteq Q$ of initial states,
- a transition function $\delta\colon Q \times \Sigma \to \mathcal{B}^+(\mathit{Comms}_Q)$,
- and a coloring $\Omega\colon Q \to \mathbb{N}$,

where $\mathcal{B}^+(\mathit{Comms}_Q)$ denotes the set of positive boolean formulas over $\mathit{Comms}_Q$. Intuitively, when the automaton is in state $q$ at position $k$ of the word $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ it guesses a set of commands $R \subseteq \mathit{Comms}_Q$ that satisfies $\delta(q, \alpha_k)$. It then spawns one copy of itself for each command in $R$ and executes the command

with that copy. If the command is of the form $(\rightarrow, q, q')$, then the corresponding copy advances to position $k+1$ and changes to state $q$. The state $q'$ is ignored. If the command is of the form $(\rightarrow_a, q, q')$ and $\alpha_k$ is a matched call, the copy jumps to the position of the matching return of $\alpha_k$ and transitions to state $q$. If $\alpha_k$ is not a matched call, then the automaton advances to position $k+1$ and transitions to state $q'$. All spawned copies of $\mathfrak{A}$ continue in parallel. A single copy of $\mathfrak{A}$ accepts $\alpha$ if the highest color visited infinitely often is even. A 1-AJA accepts $\alpha$ if all of its copies accept $\alpha$.

Formally, a run of $\mathfrak{A}$ on an infinite word $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ is a $(Q \times \mathbb{N})$-labeled tree without leaves. For a vertex $v$ we write $\mathit{succs}(v)$ to denote the successors of $v$ and for a set of vertices $V$ we write $\mathit{labels}(V)$ to denote the set of labels of $V$. For each vertex $v$ labeled with $(q, k)$, there must exist a set $R \subseteq \mathit{Comms}_Q$ such that $R \models \delta(q, \alpha_k)$ and $\mathit{labels}(\mathit{succs}(v)) = \{\mathit{app}(q, k, r) \mid r \in R\}$, where

- $\mathit{app}(q, k, (\rightarrow, q_1, q_2)) = (q_1, k+1)$,
- $\mathit{app}(q, k, (\rightarrow_a, q_1, q_2)) = (q_1, l)$ if $\alpha_k$ is a matched call and $\alpha_l$ is its matching return,
- and $\mathit{app}(q, k, (\rightarrow_a, q_1, q_2)) = (q_2, k+1)$ if $\alpha_k$ is not a matched call.

A run is initial if its root is labeled with $(q_I, 0)$ for some $q_I \in Q_I$. Each path $\pi = (q_0, k_0)(q_1, k_1)(q_2, k_2)\cdots$ through the run induces a sequence of colors $\Omega(\pi) = \Omega(q_0)\Omega(q_1)\Omega(q_2)\cdots$. A run is accepting if for all paths $\pi$ through the run the highest color occurring infinitely often in $\pi$ is even. A 1-AJA $\mathfrak{A}$ accepts a word $\alpha$ if there exists an initial accepting run of $\mathfrak{A}$ on $\alpha$.

**Lemma 4 ([4]).** *For every 1-AJA $\mathfrak{A}$ there exists an effectively constructible equivalent VPA $\mathfrak{A}_{vp}$. The size of $\mathfrak{A}_{vp}$ is exponential in the number of states of $\mathfrak{A}$.*

For a given VLDL formula $\varphi$ we now inductively construct a 1-AJA that recognizes the same language as $\varphi$. The main difficulty lies in the translation of formulas of the form $\langle\mathfrak{A}\rangle\varphi$, since these require us to translate VPAs over finite words into 1-AJAs over infinite words. We do so by adapting the idea for the translation from BVPAs to 1-AJAs from [4] and combining it with the bottom-up translation from LDL into alternating automata in [9]. The main idea of the former construction is to simulate the operations on the stack using copies of the automaton. Whenever a call occurs, the 1-AJA jumps to the matching return and guesses the state the TVPA will be in at that return. Additionally, the automaton spawns a copy of itself that verifies this guess.

**Lemma 5.** *For any VLDL formula $\varphi$ there exists an effectively constructible equivalent 1-AJA $\mathfrak{A}_\varphi$. The number of states of $\mathfrak{A}_\varphi$ is quadratic in the size of $\varphi$.*

*Proof.* We construct the automaton inductively over the structure of $\varphi$. The case $\varphi = p$ is trivial. For the boolean operations, the automata $\mathfrak{A}_\varphi$ are obtained by the closure of 1-AJAs under these operations [4].

We now consider $\varphi = \langle\mathfrak{A}\rangle\varphi'$, where $\mathfrak{A} = (Q^{\mathfrak{A}}, \widetilde{\Sigma}, \Gamma^{\mathfrak{A}}, \Delta^{\mathfrak{A}}, Q_I^{\mathfrak{A}}, F^{\mathfrak{A}}, t^{\mathfrak{A}})$. By induction we obtain a 1-AJA $\mathfrak{A}' = (Q', \widetilde{\Sigma}, \delta', Q'_I, \Omega')$ equivalent to $\varphi'$, and, for

each $\varphi_i \in \mathit{range}(t_{\mathfrak{A}})$, an equivalent 1-AJA $\mathfrak{A}_i = (Q^i, \widetilde{\Sigma}, \delta^i, Q^i_I, \Omega^i)$. We need to simulate a run of the TVPA $\mathfrak{A}$ on a finite prefix of the input $\alpha$ and, if $\mathfrak{A}$ accepts this prefix, transition into $\mathfrak{A}'$.

Consider an initial run of $\mathfrak{A}$ on such a prefix $w$, which starts with the empty stack. This run visits steps at stack height 0 finitely often. At the last visit to such a step, it either accepts the prefix it has read so far or it reads a call and moves to a step at stack height 1. In the former case, the run of $\mathfrak{A}$ has terminated. In the latter case, it visits a finite number of steps at stack height 1, and the same argument applies inductively. Note that the stack symbol pushed at the final transition from stack height 0 to stack height 1 does not influence the remainder of the run, as it is never popped nor inspected by $\mathfrak{A}$.

An example of a run on the word *clcrrcclrll* is shown in Figure 4, where $c$ is a call, $r$ is a return, and $l$ is a local action. The automaton visits the empty stack exactly once after the initial configuration, namely after reading the second $r$. Since the next symbol $c$ is a call symbol, it has to push something onto the stack, namely the symbol $A$, which is ignored in the remainder of the run. It then visits configurations with stack height 1 exactly three more times before accepting with stack height 1.

| | c | l | c | r | r | c | c | l | r | l | l | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | $q_0$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $q_5$ | $q_6$ | $q_7$ | $q_8$ | $q_9$ | $q_{10}$ | $q_{11}$ |
| | | | | $B$ | | | | $B$ | $B$ | | | |
| $\gamma$ | | $A$ | $A$ | $A$ | $A$ | | $A$ | $A$ | $A$ | $A$ | $A$ | $A$ |
| | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |

**Fig. 4.** Run of a VPA $\mathfrak{A}$ on the word *clcrrcclrll*.

The idea is to have one main copy of the automaton that jumps along the steps of the input word. Whenever this automaton encounters a call $c \in \Sigma_c$ it nondeterministically guesses whether or not the automaton eventually returns to the current stack height. If it does, then the automaton guesses some states $q'$ and $q''$ and a stack symbol $A$ such that $(q, c, q', A) \in \Delta$, it jumps to the matching return in state $q''$ and spawns a copy that verifies that it is possible to go from $q'$ to $q''$ by popping $A$ off the stack in the final transition in $\mathfrak{A}$. If the automaton never returns to the empty stack again, then the main automaton simply guesses some state $q'$ and some stack symbol $A$ with $(q, c, q', A) \in \Delta$ and moves to state $q'$. In this case it also remembers in its state space that it may not read any returns anymore, since this would falsify the guess. This is repeated until the main automaton visits an accepting state and nondeterministically decides to transition into the 1-AJA $\mathfrak{A}'$.

The verification automata work similarly to the main automaton, with the main difference that these automata do not need to guess whether or not the

automaton returns to the same stack height again upon reading a call. Since their purpose is to verify that the automaton eventually pops a symbol off the stack, they can simply assume that all pushed symbols are eventually popped as well. If they encounter an unmatched call, they enter a rejecting sink.

The run of such a 1-AJA corresponding to the run of $\mathfrak{A}$ shown in Figure 4 is shown in Figure 5. The gray line indicates the stack height, the solid black path represents the run of the main automaton, while the dashed black paths indicate the runs of the verifying automata. Dotted lines indicate that an automaton spawns a verifying automaton. For the sake of readability, the figure does not include the copies of the automata that are spawned to verify that the tests of $\mathfrak{A}$ hold true. States of the form $(q, 0)$ denote the main copy of the automaton that has not yet ignored any push actions, while states of the form $(q, 1)$ denote copies that have ignored such actions. The states $(q, q', A)$ denote verification copies that verify that it is possible to go from state $q$ to state $q'$ by popping $A$ off the stack in the final transition.

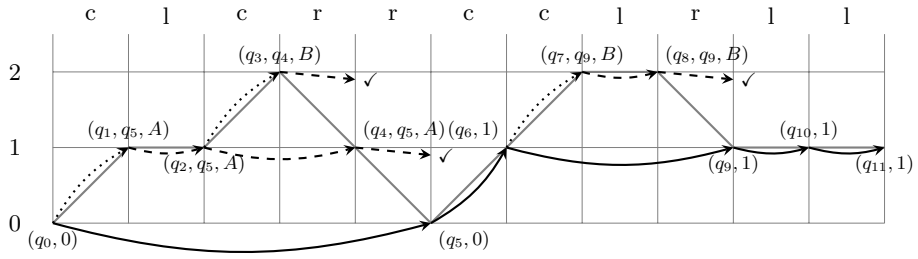


**Fig. 5.** Behavior of 1-AJA on the word *clcrrcclrll*.

Formally, we use the set of states

$$Q := (Q^{\mathfrak{A}} \times \{0,1\}) \cup (Q^{\mathfrak{A}} \times Q^{\mathfrak{A}} \times \Gamma) \cup \{rej\} \cup Q' \cup \bigcup_{\varphi_i \in range(t)} Q^i,$$

where the state *rej* is a rejecting sink, while the state sets $Q'$ and $Q^i$ are the state sets of the automata $\mathfrak{A}'$ and $\mathfrak{A}_i$, which are obtained by induction. The states from $Q^{\mathfrak{A}} \times \{0,1\}$ are used to simulate the original automaton at steps with stack height 0 and stack height at least 1, respectively.

For the sake of readability, we define the transition function for the different components of the automaton separately. We also write $(\rightarrow, q)$ and $(\rightarrow_a, q)$ as shorthands for $(\rightarrow, q, rej)$ and $(\rightarrow_a, q, rej)$, i.e., for transitions in which the latter state shall be disregarded. The easiest part of the transition function is that which controls the rejecting sink *rej*, which is defined as $\delta_{sink}(rej, a) := (\rightarrow, rej)$ for all $a \in \Sigma$. We copy the transition functions $\delta'$ and $\delta^i$ of $\mathfrak{A}'$ and $\mathfrak{A}_i$.

When encountering a final state of $\mathfrak{A}$, we need to be able to move to the successors of the initial states of $\mathfrak{A}'$ in order to model acceptance of $\mathfrak{A}$ on the

finite prefix read so far. To achieve a uniform presentation, we define the helper formula $\chi^f(q,a) := \bigvee_{q'_I \in Q'_I} \delta'(q'_I, a)$ if $q \in F^{\mathfrak{A}}$ and $\chi^f(q,a) := \mathit{false}$ otherwise.

Moreover, we need notation to denote transitions into the automata $\mathfrak{A}_i$ implementing the tests of $\mathfrak{A}$. More precisely, since we only transition into these automata upon leaving the states labeled with the respective test, we need to transition into the successors of one of the initial states of the implementing automata. To this end, we define the helper formula $\theta^a_q := \bigvee_{q_I \in Q^i_I} \delta^i(q_I, a)$, where $t(q) = \varphi_i$.

For local actions the main copy of the automaton can simply simulate the behavior of $\mathfrak{A}$ on the input word. Hence we have

$$\delta_{main}((q,b),l) := \left[\chi^f(q,l) \vee \bigvee_{(q,l,q') \in \Delta} (\rightarrow, (q',b))\right] \wedge \theta^l_q \text{ for } l \in \Sigma_l, b \in \{0,1\}$$

When reading a call, the automaton nondeterministically guesses whether it jumps to the matching return or whether it simulates the state transition while ignoring the effects on the stack. In the former case, it guesses a transition $(q,c,q',A) \in \Delta$ and a state $q'' \in Q$, spawns a verification automata verifying that it is possible to go from $q'$ to $q''$ by popping $A$ off the stack in the final transition, and continues at the matching return in state $q''$. In the latter case it ignores the effects on the stack and denotes that it may not read any returns from this point onwards by setting the binary flag in its state to 1.

$$\delta_{main}((q,b),c) := \left[\chi^f(q) \vee \bigvee_{(q,c,q',A) \in \Delta, q'' \in Q} \left[(\rightarrow, (q', q'', A)) \wedge (\rightarrow_a, (q'', b))\right] \vee \bigvee_{(q,c,q',A) \in \Delta} (\rightarrow, (q', 1))\right] \wedge \theta^c_q \text{ for } c \in \Sigma_c, b \in \{0,1\}$$

The main automaton may only handle returns as long as it has not skipped any calls. If it encounters a return after having skipped a push action, it rejects the input word, since the return falsifies its earlier guess of an unmatched call.

$$\delta_{main}((q,0),r) := \left[\chi^f(q,r) \vee \bigvee_{(q,r,\bot,q') \in \Delta} (\rightarrow, (q',0))\right] \wedge \theta^r_q \text{ for } r \in \Sigma_r$$
$$\delta_{main}((q,1),r) := (\rightarrow, \mathit{rej}) \text{ for } r \in \Sigma_r$$

The transition function $\delta_{main}$ defines the behavior of the main automaton. It remains to define the behavior of the verifying automata. These behave similarly to the main automaton on reading local actions and calls. The main difference in handling calls is that these automata do not need to guess whether or not a call is matched. Since they are only spawned on reading supposedly matched calls and accept upon reading the matching return, all calls they encounter must be matched as well. Additionally, they never transition to the automaton $\mathfrak{A}'$, but merely to the automaton implementing the respective test upon having verified

their guess.

$$\delta_{ver}((q,q',A),l) := \Big[\bigvee_{(q,l,q'')\in\Delta}(\rightarrow,(q'',q',A))\Big] \wedge \theta^l_q \text{ if } l \in \Sigma_l$$

$$\delta_{ver}((q,q',A),c) := \Big[\bigvee_{(q,c,q'',A')\in\Delta,q'''\in Q}(\rightarrow,(q'',q''',A')) \wedge (\rightarrow_a,(q''',q',A))\Big] \wedge \theta^c_q \text{ if } c \in \Sigma_c$$

$$\delta_{ver}((q,q',A),r) := \theta^r_q \text{ if } r \in \Sigma_r, (q,r,A,q') \in \Delta$$

$$\delta_{ver}((q,q',A),r) := (\rightarrow, rej) \text{ if } r \in \Sigma_r, (q,r,A,q') \notin \Delta$$

We then define the complete transition function $\delta$ of $\mathfrak{A}_\varphi$ as the union of the previously defined partial transition functions. Since their domains are pairwise disjoint, this union is well-defined.

$$\delta := \delta_{sink} \cup \delta' \cup \bigcup_{\varphi_i \in range(t)} \delta^i \cup \delta_{main} \cup \delta_{ver}$$

The coloring of $\mathfrak{A}_\varphi$ is obtained by copying the coloring of $\mathfrak{A}'$ and the $\mathfrak{A}^i$ and by coloring all states resulting from the translation of $\mathfrak{A}$ with 1. Thus, we force every path of the run of $\mathfrak{A}$ to eventually leave $\mathfrak{A}$, since this automaton only accepts a finite prefix of the input word. The 1-AJA

$$\mathfrak{A}_\varphi := (Q, \widetilde{\Sigma}, \delta, Q^{\mathfrak{A}}_I \times \{0\}, \Omega \cup \Omega' \cup \bigcup_{\varphi_i \in range(t^{\mathfrak{A}})} \Omega^i)$$

then recognizes precisely the language of $\varphi = \langle\mathfrak{A}\rangle\varphi'$, where $\Omega : q \mapsto 1$ for all $q \in (Q^{\mathfrak{A}} \times \{0,1\}) \cup (Q^{\mathfrak{A}} \times Q^{\mathfrak{A}} \times \Gamma) \cup \{rej\}$.

If $\varphi = [\mathfrak{A}]\varphi'$ we use the identity $\neg\langle\mathfrak{A}\rangle\neg\varphi' \equiv [\mathfrak{A}]\varphi'$ and construct the 1-AJA equivalent to $\neg\langle\mathfrak{A}\rangle\neg\varphi'$ instead. □

By combining Lemmas 4 and 5 we see that BVPAs are at least as expressive as VLDL formulas. This proves the direction from logic to automata of Theorem 1.

The construction via 1-AJAs yields automata of exponential size in the number of states. This blowup is unavoidable.

**Lemma 6.** *There exists a pushdown alphabet $\widetilde{\Sigma}$ such that for all $n \in \mathbb{N}$ there exists a language $L_n$ that*

- *is defined by a VLDL formula over $\widetilde{\Sigma}$ of polynomial size in $n$, and*
- *every BVPA over $\widetilde{\Sigma}$ recognizing $L_n$ has at least exponentially many states in $n$.*

*Proof.* We use the pushdown alphabet $\widetilde{\Sigma} = (\Sigma_c, \Sigma_r, \Sigma_l) = (\emptyset, \emptyset, \{0,1,\#\})$. For any $n \in \mathbb{N}$ and any $i \in [0; 2^n - 1]$ we write $\langle i\rangle_n$ to denote the binary encoding of $i$ using exactly $n$ bits. We define the singleton language $L_n := \{\#\langle 0\rangle_n\# \cdots \#\langle 2^n - 1\rangle_n\#^\omega\}$. It is known that there exists an LTL formula of polynomial length in

$n$ that defines $L_n$. Thus there also exists a VLDL formula of polynomial length defining this language due to Proposition 1.

Furthermore, since all symbols are local actions, any BVPA recognizing $L_n$ cannot use its stack and thus has to work like a traditional finite automaton with Büchi acceptance. Again, it is known that all Büchi automata recognizing $L_n$ have at least exponentially many states in $n$. Hence, all BVPAs recognizing $L_n$ have at least exponentially many states in $n$. □

## 6 Satisfiability and Validity are ExpTime-complete

We say that a VLDL formula $\varphi$ is satisfiable if it has a model. Dually, we say that $\varphi$ is valid if all words are models of $\varphi$. Instances of the satisfiability- and validity problem consist of a VLDL formula $\varphi$. The satisfiability problem asks whether or not $\varphi$ is satisfiable, whereas the validity problem asks whether or not $\varphi$ is valid. Both problems are decidable in exponential time using the construction of BVPAs from VLDL formulas from Theorem 1. We show both problems to be ExpTime-hard.

**Theorem 2.** *Both the satisfiability problem and the validity problem for VLDL are* ExpTime*-complete.*

*Proof.* Due to duality, we only show ExpTime-completeness of the satisfiability problem. Membership in ExpTime follows from closure of VLDL under negation, the membership of the emptiness-problem for 1-AJA in ExpTime [4] and Lemma 5.

We show ExpTime-hardness of the problem by a reduction from the word problem for polynomially space-bounded alternating Turing machines. Our proof is based on the reduction of the same problem to the problem of model checking pushdown systems against LTL specifications from the full version of [3]. In that reduction, an accepting run of an alternating Turing machine is encoded as a pair of a pushdown system, which checks the general format of the encoding using its stack, and an LTL specification, which checks additional properties without using a stack. We adapt this proof by checking the properties asserted by the pushdown system with a visibly pushdown automaton. Moreover, we encode the specification of the general format in the formula itself instead of splitting the specification into a pushdown system and an LTL specification.

An alternating Turing machine (ATM) [7] $\mathcal{T} = (Q_\exists, Q_\forall, \Gamma, B, q_I, \delta, F)$ consists of

- two finite disjoint sets $Q_\exists$ and $Q_\forall$ of states, which are called existential and universal states, respectively, for which we write $Q := Q_\exists \cup Q_\forall$,
- a tape alphabet $\Gamma$,
- a blank symbol $B \in \Gamma$,
- an initial state $q_I \in Q \setminus F$,
- a transition relation $\Delta \subseteq Q \times \Gamma \times Q \times \Gamma \times \{L, R\}$,
- and a set of final states $F \subseteq Q$.

We assume w.l.o.g. that every configuration has exactly two applicable transitions and that the initial state is not final.

Let $p(n)$ be some polynomial. A configuration $c$ of a $p(n)$-bounded ATM $\mathcal{T}$ on an input word $w$ is a word of length $p(|w|)+1$ over the alphabet $\Gamma \cup Q$ that contains exactly one symbol from $Q$. Let $\mathit{Conf} := \Gamma^* Q \Gamma^* \cap (Q \cup \Gamma)^{p(|w|)+1}$ denote the set of such configurations. If $c \in \mathit{Conf}$ contains a symbol from $Q_\exists$ ($Q_\forall$), we call $c$ existential (universal). Furthermore, a transition $(q, a, q', a', D) \in \Delta$ with $D \in \{L, R\}$ is existential (universal), if $q \in Q_\exists$ ($q \in Q_\forall$).

A run of a $p(n)$-bounded ATM $\mathcal{T}$ on $w$ is a tree that is labeled with configurations of $\mathcal{T}$ on $w$. Each non-terminal vertex has either one or two successors, depending on whether it is labeled with an existential or a universal configuration. These successors are labeled by one or two successor configurations. A run is accepting if all terminal vertices are labeled with final configurations. An ATM $\mathcal{T}$ accepts a word $w$ if there exists an accepting run of $\mathcal{T}$ on $w$.

An instance of the word problem consists of a $p(n)$-bounded ATM $\mathcal{T}$ and a word $w$ and asks whether or not $\mathcal{T}$ accepts $w$. This problem is ExpTime-hard [7].

We encode runs of $\mathcal{T}$ by linearizing them as words using tags of the form $<_\tau^i$ and $>_\tau^i$ for $i \in \{1, 2\}$ to delimit the encoding the first and second subtree of a vertex (recall that we assume that every configuration has at most two successors). Here, $\tau$ denotes the transition that is applied to obtain the configuration of the root of this subtree. Moreover, we use the tags $<_\ell$ and $>_\ell$ to denote leaves.

Formally, we define the pushdown alphabet $\widetilde{\Sigma} = (\Sigma_c, \Sigma_r, \Sigma_l)$ with

- $\Sigma_c = ((Q \cup \Gamma) \times \{\downarrow\}) \cup \{<_\tau^1 \mid \tau \in \Delta\} \cup \{<_\ell\}$,
- $\Sigma_r = ((Q \cup \Gamma) \times \{\uparrow\}) \cup \{>_\tau^1 \mid \tau \text{ existential}\} \cup \{>_\tau^2 \mid \tau \text{ universal}\} \cup \{>_\ell\}$, and
- $\Sigma_l = \{>_\tau^1, <_\tau^2 \mid \tau \text{ universal}\} \cup \{\#\}$.

Let $\mathit{Tags} = \{<_\tau^1, >_\tau^1 \mid \tau \text{ existential}\} \cup \{<_\tau^1, >_\tau^1, <_\tau^2, >_\tau^2 \mid \tau \text{ universal}\} \cup \{<_\ell, >_\ell\}$.

For every $w = w_0 \cdots w_n \in \Gamma^*$ and $d \in \{\downarrow, \uparrow\}$, let $(w, d) := (w_0, d) \cdots (w_n, d)$, which we lift to languages in the obvious way. Furthermore, $w^r := w_n \cdots w_0$. Let $c \in \mathit{Conf}$. We define $\mathit{push}(c) := (c, \downarrow)$ and $\mathit{pop}(c) := (c^r, \uparrow)$

Using this, we encode a run of $\mathcal{T}$ by recursively iterating over its vertices $v$ as follows:

- $\mathit{enc}(v) := <_\ell \cdot \mathit{push}(c) \cdot >_\ell \cdot \mathit{pop}(c)$, if $v$ is a leaf labeled with the configuration $c$.
- $\mathit{enc}(v) := <_\tau^1 \cdot \mathit{push}(c) \cdot \mathit{enc}(v_1) \cdot >_\tau^1 \cdot \mathit{pop}(c)$, if $v$ has a single child $v_1$, $v$ is labeled by the (existential) configuration $c$, and $\tau$ is the transition that is applied to $c$ to obtain the label of $v_1$.
- $\mathit{enc}(v) := <_{\tau_1}^1 \cdot \mathit{push}(c) \cdot \mathit{enc}(v_1) \cdot >_{\tau_1}^1 \cdot \mathit{pop}(c) <_{\tau_2}^2 \cdot \mathit{push}(c) \cdot \mathit{enc}(v_2) \cdot >_{\tau_2}^2 \cdot \mathit{pop}(c)$, if $v$ has two children $v_1$ and $v_2$, $v$ is labeled by the (universal) configuration $c$, and $\tau_i$, for $i \in \{1, 2\}$, is the transition that is applied to $c$ to obtain the label of $v_i$.

Thus, a complete run with root $v$ is encoded by $\mathit{enc}(v) \cdot \#^\omega$. Our goal is to construct a formula $\varphi_{\mathcal{T},w}$ that is satisfied only by words that encode initial accepting runs of $\mathcal{T}$ on $w$. To this end, we need to formalize the following six conditions on an infinite word $\alpha \in \Sigma^\omega$:

1. $\alpha \in (\mathit{Tags} \cdot \mathit{Conf})^+ \cdot \#^\omega$ and begins with $<_\tau^1 \cdot (c_I, \downarrow)$, where $c_I$ is the initial configuration of $\mathcal{T}$ on $w$ and where $\tau$ is a transition that is applicable to $c_I$.
2. Every $<_\tau^i$, $i \in \{1, 2\}$, is directly followed by $(c, \downarrow)$ for some configuration $c$ to which $\tau$ is applicable. Furthermore, say the stack height is $n$ after this infix. Then, we require that this stack height is reached again at a later position, and at the first such position, the infix $>_\tau^1 \cdot (c^r, \uparrow)$ starts.
3. Every $>_\tau^1$ with universal $\tau$, which is directly followed by $(c^r, \uparrow)$ for some configuration $c$ (assuming the previous condition is satisfied), is directly followed by $(c^r, \uparrow) \cdot <_{\tau'}^2 \cdot (c, \downarrow)$, where $\tau' \neq \tau$ is the unique other transition that is applicable to $c$.
4. Every $<_\tau^i$, $i \in \{1, 2\}$, is directly followed by $(c, \downarrow) < (c', \downarrow)$ for some $< \in \{<_\tau^1 | \tau \in \Delta\} \cup \{<_\ell\}$ such that $\tau$ is applicable to $c$ and $c'$ is the corresponding successor configuration.
5. Every $<_\ell$ is directly followed by $(c, \downarrow) >_\ell (c^r, \uparrow)$ for some accepting configuration of $\mathcal{T}$.
6. Stack height zero has to be reached after a non-empty prefix, and from the first such position onwards, only # appears.

It is straightforward to come up with polynomially-sized VLDL formulas expressing these conditions (note that only the second and sixth condition require non-trivial usage of the stack). Furthermore, $\alpha$ satisfies the conjunction of these properties if, and only if, it encodes an accepting run of $\mathcal{T}$ on $w$. Thus, as the word problem for polynomially space-bounded ATMs is ExpTime-hard, the satisfiability problem for VLDL is ExpTime-hard as well. □

## 7 Model Checking is ExpTime-complete

We now consider the model checking problem for VLDL. Given a VPS $\mathcal{S}$ and an initial state $q_I$ of $\mathcal{S}$, we define $\mathit{traces}(\mathcal{S})$ as the set of all infinite words $\alpha$ for which there exists a run of $\mathcal{S}$ on $\alpha$ that starts in $q_I$. An instance of the model checking problem consists of a VPS $\mathcal{S}$, an initial state $q_I$ of $\mathcal{S}$, and a VLDL formula $\varphi$ and asks whether $\mathit{traces}(\mathcal{S}) \subseteq L(\varphi)$ holds true. We call $\mathcal{S}$ the system and $\varphi$ the specification.

This problem is decidable in exponential time using the quadratic-size construction of 1-AJA from the proof of Lemma 5 and an exponential-time model checking algorithm for 1-AJA [4]. Moreover, the problem is ExpTime-hard.

**Theorem 3.** *Model checking VLDL specifications against VPS is* ExpTime*-complete.*

*Proof.* Membership in ExpTime follows from Lemma 5 and the membership of the problem of checking visibly pushdown systems against 1-AJA specifications in ExpTime [4]. Moreover, since the validity problem for VLDL is ExpTime-hard and since validity of $\varphi$ is equivalent to $\mathit{traces}(\mathcal{S}_{univ}) \subseteq \varphi$, where $\mathcal{S}_{univ}$ with $\mathit{traces}(\mathcal{S}_{univ}) = \Sigma^\omega$ is effectively constructible in constant time, the model checking problem for VLDL is ExpTime-hard as well. □

## 8 Solving VLDL Games is 3ExpTime-complete

In this section we investigate visibly pushdown games with winning conditions given by VLDL formulas. We consider games with two players, which we call the input- and the output player, or $P_I$ and $P_O$, respectively. We assume that the set $P$ of atomic propositions is partitioned into disjoint subsets $I$ and $O$, which we call the input- and output propositions. This partition of $P$ is independent of the partition of $2^P$ into a pushdown alphabet $\widetilde{\Sigma}$.

A two-player game with VLDL winning condition $\mathcal{G} = (V_I, V_O, \Sigma, E, v_I, \ell, \varphi)$ consists of

- two disjoint, possibly countably infinite sets $V_I$ and $V_O$ of vertices,
- a finite alphabet $\Sigma$,
- an initial state $v_I \in V_I \cup V_O$,
- a set of edges $E \subseteq (V_I \cup V_O)^2$,
- a state-labeling $\ell\colon (V_I \cup V_O) \to \Sigma$,
- and a VLDL formula $\varphi$, which we call the winning condition.

A play $\pi = v_0 v_1 v_2 \cdots$ of $\mathcal{G}$ is a infinite sequence of vertices of $\mathcal{G}$ with $(v_i, v_{i+1}) \in E$ for all $i \geq 0$. We call $\pi$ initial if $v_0 = v_I$. The play $\pi$ is winning for $P_O$ if the infinite word $\ell(v_1)\ell(v_2)\ell(v_3)\cdots$ [2] is a model of $\varphi$. Otherwise $\pi$ is winning for $P_I$.

A strategy for $P_X$ is a function $\sigma\colon V^*V_X \to V_I \cup V_O$, such that $(v, \sigma(w \cdot v) \in E$ for all $v \in V_X$, $w \in V^*$. We call a play $\pi = v_0 v_1 v_2 \cdots$ consistent with $\sigma$ if for all finite prefixes $\pi' = v_0 \cdots v_n$ of $\pi$ with $v_n \in V_X$ we have $\sigma(\pi') = v_{n+1}$. A strategy $\sigma$ is winning for $P_X$ if all initial plays that are consistent with $\sigma$ are winning for $P_X$. We say that the input- or the output player wins $\mathcal{G}$ if he or she has a winning strategy. If either $P_I$ or $P_O$ wins a given game $\mathcal{G}$, we say that $\mathcal{G}$ is determined.

A visibly pushdown game (VPG) with a VLDL winning condition $\mathcal{H} = (\mathcal{S}, Q_I, Q_O, q_I, \varphi)$ consists of

- a VPS $\mathcal{S} = (Q, \widetilde{\Sigma}, \Gamma, \Delta)$,
- a partition of $Q$ into $Q_I$ and $Q_O$,
- an initial state $q_I \in Q$,
- and a VLDL formula $\varphi$.

The VPG $\mathcal{H}$ then defines the two-player game $\mathcal{G}_\mathcal{H} = (V_I, V_O, \Sigma, E, v_I, \ell, \varphi)$ with

- $V_X := Q_X \times ((\Gamma \setminus \{\bot\})^* \cdot \bot) \times \Sigma$,
- $v_I = (q_I, \bot, a)$ for some $a \in \Sigma$ (recall that the trace that determines the winner of a play disregards the label of the initial vertex),
- $((q, \gamma, a), (q', \gamma', a')) \in E$ if and only there is an $a'$-labeled edge from $(q, \gamma)$ to $(q, \gamma')$ in the configuration graph $G_S$,
- and $\ell\colon (q, \gamma, a) \mapsto a$.

Solving a VPG $\mathcal{H}$ means deciding whether or not $P_O$ wins $\mathcal{G}_\mathcal{H}$.

[2] Note that the trace starts with $v_1$.

**Proposition 3.** *VPGs with VLDL winning conditions are determined.*

*Proof.* Since each VLDL formula defines a language in $\omega$-VPL, each VPG with a VLDL winning condition is equivalent to a VPG with an $\omega$-VPL winning condition, which are known to be determined [12]. □

It is possible to solve VPGs with VLDL winning conditions by constructing a VPA $\mathfrak{A}_\varphi$ from the winning condition $\varphi$ and then solving the visibly pushdown game with a VPA winning condition using the method from [12]. This approach runs in triply-exponential time in $|\varphi|$ and exponential time in $|\mathcal{S}|$. Moreover, the problem is 3ExpTime-hard.

**Theorem 4.** *Solving VPGs with VLDL winning conditions is* 3ExpTime*-complete.*

*Proof.* Membership in 3ExpTime follows from the membership of solving VPGs against VPA winning conditions in 2ExpTime [12] and Lemma 5.

We show 3ExpTime-hardness of the problem with a reduction from solving pushdown games with LTL winning conditions. A pushdown game with an LTL winning condition $\mathcal{H} = (\mathcal{S}, V_I, V_O, \psi)$ is defined similarly to a VPG, except for the relaxation that $\mathcal{S}$ may now be a traditional pushdown system instead of a visibly pushdown system. Specifically, we have $\Delta \subseteq (Q \times \Gamma \times \Sigma \times Q \times \Gamma^{\leq 2})$, where $\Gamma^{\leq 2}$ denotes the set of all words over $\Gamma$ of at most two letters. Stack symbols are popped off the stack using transitions of the form $(q, A, a, q', \varepsilon)$, the top of the stack can be tested and changed with transitions of the form $(q, A, a, q', B)$, and pushes are realized with transitions of the form $(q, A, a, q', BC)$. Additionally, the winning condition is given as an LTL formula instead of a VLDL formula. The two-player game $\mathcal{G}_\mathcal{H}$ is defined analogously to the visibly pushdown game.

Since the pushdown game admits transitions such as $(q, A, a, q', BC)$, which pop $A$ off the stack and push $B$ and $C$ onto it, we need to split such transitions into several transitions in the visibly pushdown game. We modify the original game such that every transition of the original game is modeled by three transitions in the visibly pushdown game, up to two of which may be dummy actions that do not change the stack. As each transition may perform at most three operations on the stack, we can keep track of the list of changes still to be performed in the state space. We perform these actions using dummy letters $c$ and $l$, which we add to $\Sigma$ and read while performing the required actions on the stack. We choose the vertices $V'_X = V_X \cup (V_X \times (\Gamma \cup \{\#\})^{\leq 2})$ and the alphabet $\widetilde{\Sigma} = (\{c\}, \Sigma, \{l\})$.

We transform $\mathcal{H}$ as shown in Figure 6 and obtain the VPG $\mathcal{H}'$. Moreover, we transform $\psi$ into $\psi'$ by inductively replacing each occurrence of $\mathbf{X}\psi$ by $\mathbf{X}^3\psi'$ and each occurrence of $\psi_1 \mathbf{U} \psi_2$ by $(\psi'_1 \vee c \vee l)\mathbf{U}(\psi'_2 \wedge \neg c \wedge \neg l)$. We subsequently translate the obtained LTL formula $\psi'$ into an equivalent VLDL formula $\varphi$ using Proposition 1. The input player wins $\mathcal{H}'$ with the winning condition $\varphi$ if and only he wins $\mathcal{H}$ with the winning condition $\psi$. Hence, solving VPGs with VLDL winning conditions is 3ExpTime-hard. □

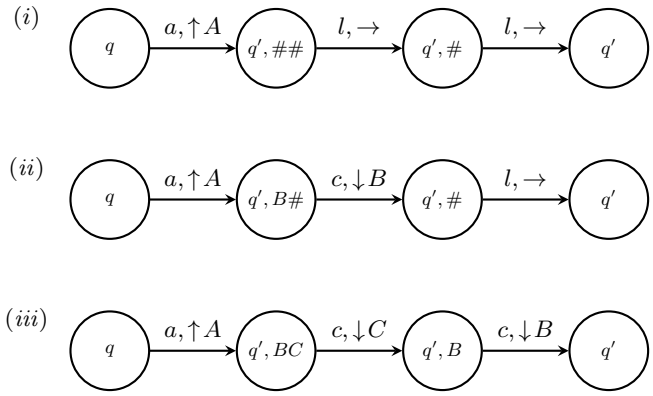


**Fig. 6.** Construction of a VPG from a pushdown game for transitions of the forms $(i)$ $(q, a, A, q', \varepsilon)$, $(ii)$ $(q, a, A, q', B)$, and $(iii)$ $(q, a, A, q', BC)$.

## 9 Pushdown Linear Dynamic Logic

In this work we extend LDL by replacing the regular expressions used as guards for the temporal operators by VPAs. The next step would be to replace the VPA by a more powerful automaton model, for example deterministic pushdown automata (DPDA). However, all interesting decision problems for the resulting logic called Deterministic Pushdown Linear Dynamic Logic (DPLDL) are undecidable.

**Theorem 5.** *The satisfiability problem for DPLDL is undecidable.*

*Proof.* We show the undecidability using a reduction from the problem of deciding nonemptiness of the intersection of two DPDA, which is known to be undecidable [10]. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two DPDA over a shared alphabet $\Sigma$, pick $\# \notin \Sigma$ and consider $\varphi := \langle \mathfrak{A}_1 \rangle \# \wedge \langle \mathfrak{A}_2 \rangle \#$. Then $\varphi$ is satisfiable if and only if $L(\mathfrak{A}_1) \cap L(\mathfrak{A}_2) \neq \emptyset$, whence satisfiability of DPLDL is undecidable. □

Since the satisfiability problem reduces to the model checking and the problem of solving pushdown games against DPLDL winning conditions, these problems are undecidable as well.

**Corollary 1.** *The validity problem and the model checking problem for DPLDL as well as the problem of solving pushdown games against DPLDL winning conditions are undecidable.*

Since every DPDA is also a PDA, the extension of DPLDL by nondeterministic pushdown automata inherits these undecidability results from DPLDL. Thus, VLDL is, to the best of our knowledge, the most expressive logic that combines the temporal modalities of LDL with guards specified by classical automata models and still has decidable decision problems.

## 10 Conclusion

We have introduced Visibly Linear Dynamic Logic (VLDL) which strengthens Linear Dynamic Logic (LDL) by replacing the regular expressions used as guards in the latter logic with visibly pushdown automata. VLDL is as expressive as the visibly pushdown languages. We have provided effective translations from VLDL into BVPA and vice versa with an exponential blowup in size in both directions. From automata to logic, this blowup cannot be avoided. It remains open whether or not the exponential blowup in the other direction can be avoided.

Figure 7 gives an overview over the known formalisms that capture $\omega$-VPL and the translations between them. The constructions described in this work are marked by solid lines, all others by dotted lines.

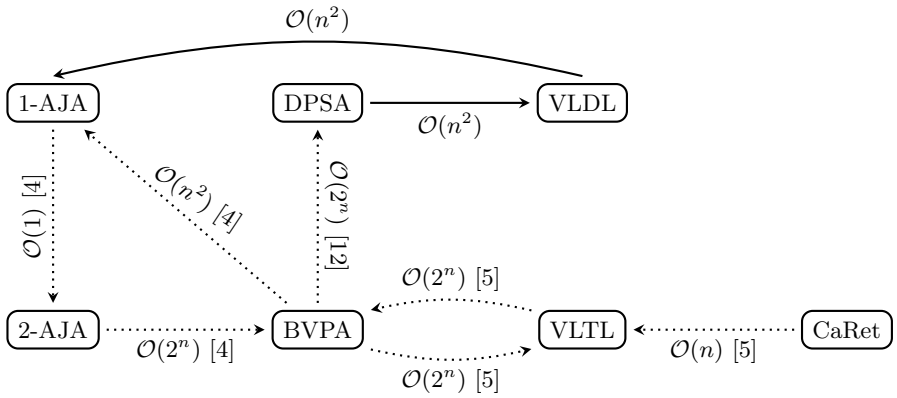


**Fig. 7.** Formalisms capturing (subsets of) $\omega$-VPL and translations between them.

We have shown that the satisfiability problem and the emptiness problem for VLDL are ExpTime-complete. Model checking visibly pushdown systems against VLDL specifications is ExpTime-complete as well. Moreover, we proved that solving visibly pushdown games with VLDL winning conditions is 3ExpTime-complete.

Additionally, we have shown that removing the restriction that the guards are visibly pushdown automata and replacing them even with deterministic pushdown automata yields a logic with an undecidable satisfiability problem. It remains an open question whether there exist formalisms with expressive power inbetween visibly pushdown automata and deterministic pushdown automata that yield similar logics with decidable decision problems.

In contrast to LDL [15] and VLTL [5], we have used automata to define guards instead of regular or rational expressions. Even though the visibly rational expressions (VRE) [6] used in VLTL can be translated into VPAs of quadratic size, it is not clear how to translate the past-operators of VLTL into

VLDL directly. There exist translations between VLTL and VLDL that incur a doubly-exponential blowup in both directions, using a detour over BVPAs, as shown in Figure 7. In spite of this blowup in the translation from automata to regular expressions, the satisfiability problem and the model checking problem for both logics are ExpTime-complete. It remains open whether there exist efficient translations between the two logics.

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