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Automata, Games and Verification: Lecture 2

3 ω-regular Languages

Definition 1 *The* ω-regular expressions *are defined as follows.*

- *If R is an regular expression where $\epsilon \notin \mathcal{L}(R)$, then R^ω is an ω-regular expression.*
  $\mathcal{L}(R^\omega) = \mathcal{L}(R)^\omega$
  where $L^\omega = \{u_0 u_1 \ldots \mid u_i \in L, |u_i| > 0$ for all $i \in \omega\}$ for $L \subseteq \Sigma^*$.
- *If R is a regular expression and U is an ω-regular expression, then $R \cdot U$ is an ω-regular expression.*
  $\mathcal{L}(R \cdot U) = \mathcal{L}(R) \cdot \mathcal{L}(U)$
  where $L_1 \cdot L_2 = \{r \cdot u \mid r \in L_1, u \in L_2\}$ for $L_1 \subseteq \Sigma^*, L_2 \subseteq \Sigma^\omega$.
- *If U_1 and U_2 are ω-regular expressions, then $U_1 + U_2$ is an ω-regular expression.*
  $\mathcal{L}(U_1 + U_2) = \mathcal{L}(U_1) \cup \mathcal{L}(U_2)$.

Definition 2 *An* ω-regular *language is a finite union of ω-languages of the form $U \cdot V^\omega$ where $U, V \subseteq \Sigma^*$ are regular languages.*

Theorem 1 *If L_1 and L_2 are Büchi recognizable, then so is $L_1 \cup L_2$.*

Proof:

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Büchi automata that recognize L_1 and L_2, respectively. We construct an automaton $\mathcal{A}'$ for $L_1 \cup L_2$:

- $S' = S_1 \cup S_2$ (w.l.o.g. we assume $S_1 \cap S_2 = \emptyset$);
- $I' = I_1 \cup I_2$;
- $T' = T_1 \cup T_2$;
- $F' = F_1 \cup F_2$.

$\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$: For $\alpha \in \mathcal{L}(A')$, we have an accepting run $r = s_0 s_1 s_2 \ldots$ of α in $\mathcal{A}'$. If $s_0 \in S_1$, then r is an accepting run on $\mathcal{A}_1$, otherwise $s_0 \in S_2$ and r is an accepting run on $\mathcal{A}_2$.

$\mathcal{L}(\mathcal{A}') \supseteq \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$: For $i \in \{1, 2\}$ and $\alpha \in \mathcal{L}(\mathcal{A}_i)$, there is an accepting run $r = s_0 s_1 s_2 \ldots$ on $\mathcal{A}_i$. The run r is accepting for α in $\mathcal{A}'$. ■

Theorem 2 *If L_1 and L_2 are Büchi recognizable, then so is $L_1 \cap L_2$.*

Proof:

We construct an automaton $\mathcal{A}'$ from $\mathcal{A}_1$ and $\mathcal{A}_2$:

- $S' = S_1 \times S_2 \times \{1, 2\}$
- $I' = I_1 \times I_2 \times \{1\}$
- $T' = \{((s_1, s_2, 1), \sigma, (s'_1, s'_2, 1)) \mid (s_1, \sigma, s'_1) \in T_1, (s_2, \sigma, s'_2) \in T_2, s_1 \notin F_1\}$
  $\cup \{((s_1, s_2, 1), \sigma, (s'_1, s'_2, 2)) \mid (s_1, \sigma, s'_1) \in T_1, (s_2, \sigma, s'_2) \in T_2, s_1 \in F_1\}$
  $\cup \{((s_1, s_2, 2), \sigma, (s'_1, s'_2, 2)) \mid (s_1, \sigma, s'_1) \in T_1, (s_2, \sigma, s'_2) \in T_2, s_2 \notin F_2\}$
  $\cup \{((s_1, s_2, 2), \sigma, (s'_1, s'_2, 2)) \mid (s_1, \sigma, s'_1) \in T_1, (s_2, \sigma, s'_2) \in T_2, s_2 \in F_2\}$
- $F' = \{(s_1, s_2, 2) \mid s_1 \in S_1, s_2 \in F_2\}$

$\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$:

- $r' = (s_1^0, s_2^0, t^0)(s_1^1, s_2^1, t^1)\ldots$ is a run of $\mathcal{A}'$ on input word σ iff $r_1 = s_1^0 s_1^1 \ldots$ is a run of $\mathcal{A}_1$ on σ and $r_2 = s_2^0 s_2^1 \ldots$ is a run of $\mathcal{A}_2$ on σ.
- r is accepting iff r_1 is accepting and r_2 is accepting.

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Theorem 3 *If L_1 is a regular language and L_2 is Büchi recognizable, then $L_1 \cdot L_2$ is Büchi-recognizable.*

Proof:

Let $\mathcal{A}_1$ be a finite-word automaton that recognizes L_1 and $\mathcal{A}_2$ be a Büchi automaton that recognizes L_2. We construct:

- $S' = S_1 \cup S_2$ (w.l.o.g. we assume $S_1 \cap S_2 = \emptyset$;
- $I' = \begin{cases} I_1 & \text{if } I_1 \cap F_1 = \emptyset \\ I_1 \cup I_2 & \text{otherwise;} \end{cases}$
- $T' = T_1 \cup T_2 \cup \{(s, \sigma, s') \mid (s, \sigma, f) \in T_1, f \in F_1, s' \in I_2\}$;
- $F' = F_2$.

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Theorem 4 *If L is a regular language then L^ω is Büchi recognizable.*

Proof:

Let $\mathcal{A}$ be a finite word automaton; let w.l.o.g. $\epsilon \notin \mathcal{L}(\mathcal{A})$.

- **Step 1:** Ensure that all initial states have no incoming transitions. We modify $\mathcal{A}$ as follows:
  - $S' = S \cup \{s_{\text{new}}\}$;
  - $I' = \{s_{\text{new}}\}$;
  - $T' = T \cup \{(s_{\text{new}}, \sigma, s') \mid (s, \sigma, s') \in T \text{ for some } s \in I\}$;

– $F' = F$.

This modification does not affect the language of $\mathcal{A}$.

- **Step 2:** Add loop:
  - $S'' = S'$; $I'' = I'$;
  - $T'' = T' \cup \{(s, \sigma, s_{\text{new}} \mid (s, \sigma, s') \in T'$ and $s' \in F'\}$;
  - $F'' = I'$.

$\mathcal{L}(\mathcal{A}'') \subseteq \mathcal{L}(\mathcal{A}')^\omega$:

- Assume $\alpha \in \mathcal{L}(\mathcal{A}'')$ and $s_0 s_1 s_2 \ldots$ is an accepting run for α in $\mathcal{A}''$.
- Hence, $s_i = s_{\text{new}} \in F'' = I'$ for infinitely many indices i: $i_0, i_1, i_2, \ldots$.
- This provides a series of runs in $\mathcal{A}'$:
  - run $s_0 s_1 \ldots s_{i_1-1} s$ on $w_1 = \alpha(0)\alpha(1)\ldots\alpha(i_1 - 1)$ for some $s \in F'$;
  - run $s_{i_1} s_{i_1+1} \ldots s_{i_2-1} s$ on $w_2 = \alpha(i_1)\alpha(i_1 + 1)\ldots\alpha(i_2 - 1)$ for some $s \in F'$;
  - ...
- This yields $w_k \in \mathcal{L}(\mathcal{A}')$ for every $k \geq 1$.
- Hence, $\alpha \in \mathcal{L}(\mathcal{A}')^\omega$.

$\mathcal{L}(\mathcal{A}'') \supseteq \mathcal{L}(\mathcal{A}')^\omega$:

- Let $\alpha = w_1 w_2 w_3 \in \Sigma^\omega$ such that $w_k \in \mathcal{L}(\mathcal{A}')$ for all $k \geq 1$.
- For each k, we choose an accepting run $s_0^k s_1^k s_2^k \ldots s_{n_k}^k$ of $\mathcal{A}'$ on w_k.
- Hence, $s_0^k \in I'$ and $s_{n_k}^k \in F'$ for all $k \geq 1$.
- Thus,
$$s_0^1 \ldots s_{n_1-1}^1 s_0^2 \ldots s_{n_2-1}^2 s_0^3 \ldots s_{n_3-1}^3 \ldots$$
  is an accepting run on α in $\mathcal{A}''$.
- Hence, $\alpha \in \mathcal{L}(\mathcal{A}'')$.

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Theorem 5 (Büchi's Characterization Theorem (1962)) *An ω-language is Büchi recognizable iff it is ω-regular.*

Proof:

"$\Leftarrow$" follows from previous constructions.

"$\Rightarrow$": Given a Büchi automaton $\mathcal{A}$, we consider for each pair $s, s' \in S$ the regular language

$$W_{s,s'} = \{u \in \Sigma^* \mid \text{finite-word automaton } (S, \{s\}, T, \{s'\}) \text{ accepts } u\ \} .$$

Claim: $\mathcal{L}(\mathcal{A}) = \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot {W_{s',s'}}^\omega$.

$\mathcal{L}(\mathcal{A}) \subseteq \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot {W_{s',s'}}^\omega$:

- Let $\alpha \in \mathcal{L}(\mathcal{A})$.
- Then there is an accepting run r for α on $\mathcal{A}$, which begins at some $s \in I$ and visits some $s' \in F$ infinitely often:

$$r : s \xrightarrow{\alpha_0} s' \xrightarrow{\alpha_1} s' \xrightarrow{\alpha_2} s' \xrightarrow{\alpha_3} s' \xrightarrow{\alpha_4} s' \rightarrow \ \dots,$$

  where $\alpha = \alpha_0 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \cdot \dots$.
  (Notation:
  $s_0 \overset{\sigma_0\sigma_1,\dots\sigma_k}{\longrightarrow} s_{k+1}$: there exist $s_1, \dots s_k$ s.t. $(s_i, \sigma_i, s_{i+1}) \in$ for all $0 \leq i \leq k$.)
- Hence, $\alpha_0 \in W_{s,s'}$ and $\alpha_k \in W_{s',s'}$ for $k > 0$ and thus $\alpha \in W_{s,s'} \cdot W_{s',s'}{}^{\omega}$ for some $s \in I, s' \in F$.

$\mathcal{L}(\mathcal{A}) \supseteq \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot W_{s',s'}{}^{\omega}$:

- Let $\alpha = \alpha_0 \cdot \alpha_1 \cdot \alpha_2 \cdot \dots$ with $\alpha_0 \in W_{s,s'}$ and $\alpha_k \in W_{s',s'}$ for some $s \in I, s' \in F$.
- Then the run

$$r : s \xrightarrow{\alpha_0} s' \xrightarrow{\alpha_1} s' \xrightarrow{\alpha_2} s' \xrightarrow{\alpha_3} s' \xrightarrow{\alpha_4} s' \rightarrow$$

  exists and is accepting since $s' \in F$.
- It follows that $\alpha \in \mathcal{L}(\mathcal{A})$.

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4 Deterministic Büchi Automata

Theorem 6 *The language $L = \{\alpha \in \Sigma^\omega \mid In(\alpha) = \{b\}\}$ over $\Sigma = \{a, b\}$ is not recognizable by a deterministic Büchi automaton.*

Proof:

- Assume that L is recognized by the deterministic Büchi automaton $\mathcal{A}$.
- Since $b^\omega \in L$, there is a run
  $r_0 = s_{0,0}s_{0,1}s_{0,2}, \dots$
  with $s_{0,n_0} \in F$ for some $n_0 \in \omega$.
- Similarly, $b^{n_0}ab^\omega \in L$ and there must be a run
  $r_1 = s_{0,0}s_{0,1}s_{0,2} \dots s_{0,n_0}s_1s_{1,0}s_{1,1}s_{1,2} \dots$
  with $s_{1,n_1} \in F$
- Repeating this argument, there is a word
  $b^{n_0}ab^{n_1}ab^{n_2}a \dots$
  accepted by $\mathcal{A}$.
- This contradicts $L = \mathcal{L}(\mathcal{A})$.

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