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Automata, Games and Verification: Lecture 8

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11 Weak Monadic Second-Order Theory of One Successor (WS1S)

Syntax: same as S1S;

Semantics: same as S1S; except:
$\sigma_1, \sigma_2 \models \exists X.\varphi$ iff there is a **finite** $A \subseteq \omega$ s.t.

$$\sigma_2'(X) = \begin{cases} \sigma_2(X) \text{ if } X \neq X_i \\ A \text{ otherwise} \end{cases}$$

and $\sigma_1, \sigma_2' \models \varphi$.

Theorem 1 *A language is WS1S-definable iff it is S1S-definable.*

Proof:

($\Rightarrow$): Quantifier relativization:

$$\begin{array}{lcl} \forall X \ldots & \mapsto & \forall X.\ \mathrm{Fin}(X) \rightarrow \ldots \\ \exists X \ldots & \mapsto & \forall X.\ \mathrm{Fin}(X) \wedge \ldots \end{array}$$

($\Leftarrow$):

- Let φ be an S1S-formula.
- Let $\mathcal{A}$ be a Büchi automaton with $\mathcal{L}(\mathcal{A}) = \mathrm{models}(\varphi)$.
- Let $\mathcal{A}'$ be a deterministic Muller automaton with $\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A})$..
- By the characterization of deterministic Muller languages, $\mathcal{L}(\mathcal{A}')$ is a boolean combination of languages $\overrightarrow{W}$, where W is finite-word recognizable.
- Let $\psi(y)$ be a WS1S formula that defines the words whose prefix up to position y is in W.
- $\varphi' := \forall x.\ \exists y.\ (x < y \wedge \psi(y))$.

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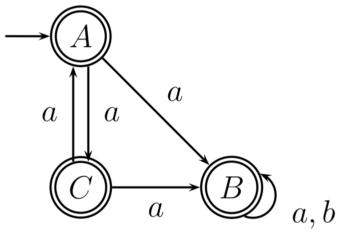
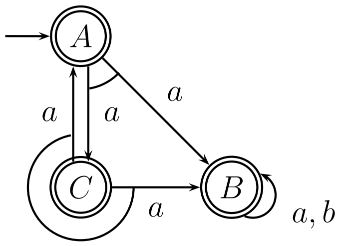
12 Alternating Automata

Example:

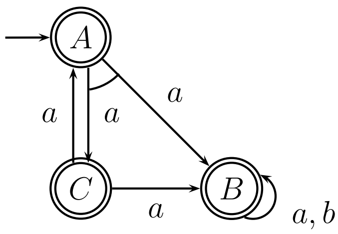
- Nondeterministic automaton, $L = a(a+b)^\omega$, existential branching mode:



- $\forall$-automaton, $L = a^\omega$, universal branching mode:



- Alternating automaton, both branching modes (arc between edges indicates universal branching mode), $L = aa(a+b)^\omega$



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Definition 1 *The* positive Boolean formulas *over a set X, denoted $\mathbb{B}^+(X)$, are the formulas built from elements of X, conjunction $\wedge$, disjunction $\vee$, true and false.*

Definition 2 *A set $Y \subseteq X$* satisfies *a formula $\varphi \in B^+(X)$, denoted $Y \models \varphi$, iff the truth assignment that assigns true to the members of Y and false to the members of $X \setminus Y$ satisfies φ.*

Definition 3 *An* alternating Büchi automaton *is a tuple $\mathcal{A} = (S, s_0, \delta, F)$, where:*

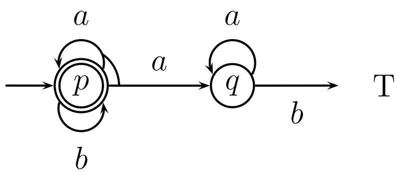
- *S is a finite set of states,*
- *$s_0 \in S$ is the initial state,*
- *$F \subseteq S$ is the set of accepting states, and*
- *$\delta : S \times \Sigma \rightarrow \mathbb{B}^+(S)$ is the transition function.*

A tree T over a set of *directions* D is a prefix-closed subset of D^*. The empty sequence ϵ is called the *root*. The children of a node $n \in T$ are the nodes children$(n) = \{n \cdot d \in T \mid d \in D\}$. A Σ-labeled tree is a pair (T, l), where $l : T \to \Sigma$ is the labeling function.

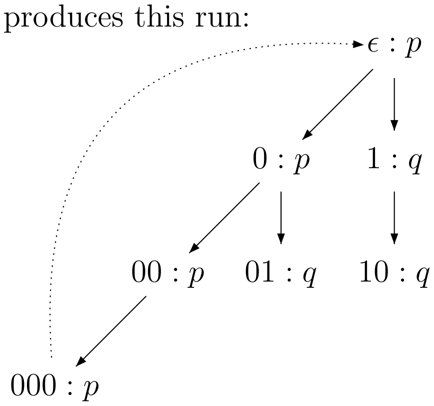
Definition 4 *A* run *of an alternating automaton on a word* $\alpha \in \Sigma^\omega$ *is an* S*-labeled* tree $\langle T, r\rangle$ *with the following properties:*

- $r(\epsilon) = s_0$ *and*
- *for all* $n \in T$*, if* $r(n) = s$*, then* $\{r(n') \mid n' \in \mathit{children}(n)\}$ *satisfies* $\delta(s, \alpha(|n|))$*.*

Example: $L = (\{a, b\}^* \, b)^\omega$



$S = \{p, q\}$
$F = \{p\}$
$\delta(p, a) = p \wedge q$
$\delta(p, b) = p$
$\delta(q, a) = q$
$\delta(q, b) = \mathrm{T}$
example word $w = (aab)^\omega$ produces this run:



(the dotted line means that the same tree would repeat there) ✚

Definition 5 *A* branch *of a tree* T *is a maximal sequence of words* $n_0, n_1, n_2, \cdots$ *such that* $n_0 = \epsilon$ *and* n_{i+1} *is a child of* n_i *for* $i \geq 0$*.*

Notation: Infinity set of a branch β in a run tree (T, r):

$$In(\beta) = \{s \in S \mid \forall i \exists j : j \geq i \wedge r(\beta(j)) = s\}$$

Definition 6 *A run* (T, r) *is* accepting *iff, for every infinite branch* $Y' \subseteq Y$*,*

$$In(Y') \cap F \neq \emptyset.$$

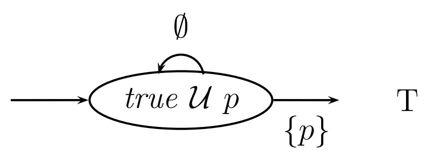
Theorem 2 *For every LTL formula* φ*, there is an alternating Büchi automaton* $\mathcal{A}$ *with* $\mathcal{L}(\mathcal{A}) = \mathit{models}(\varphi)$

Proof:

- $S = \text{closure}(\varphi) := \{\psi, \neg\psi \mid \psi \text{ is subformula of } \varphi\}$;
- $s_0 = \varphi$;
- $\delta(p,a) = \mathit{true}$ if $p \in a$, false if $p \notin a$;
  $\delta(\neg p,a) = \mathit{false}$ if $p \in a$, true if $p \notin a$;
  $\delta(\mathit{true},a) = \mathit{true}$;
  $\delta(\mathit{false},a) = \mathit{false}$;
- $\delta(\psi_1 \wedge \psi_2, a) = \delta(\psi_1,a) \wedge \delta(\psi_2,a)$;
- $\delta(\psi_1 \vee \psi_2, a) = \delta(\psi_1,a) \vee \delta(\psi_2,a)$;
- $\delta(\bigcirc\, \psi, a) = \psi$;
- $\delta(\psi_1 \,\mathcal{U}\, \psi_2, a) = \delta(\psi_1,a) \vee (\delta(\psi_2,a) \wedge \psi_1 \,\mathcal{U}\, \psi_2)$;
- $\delta(\neg\psi, a) = \overline{\delta(\psi,a)}$;
- $\overline{\psi} = \neg\psi$ for $\psi \in S$;
- $\overline{\neg\psi} = \psi$ for $\psi \in S$;
- $\overline{\psi_1 \wedge \psi_2} = \overline{\alpha} \vee \overline{\beta}$;
- $\overline{\psi_1 \vee \psi_2} = \overline{\alpha} \wedge \overline{\beta}$;
- $\overline{\mathit{true}} = \mathit{false}$;
- $\overline{\mathit{false}} = \mathit{true}$;
- $F = \{\neg(\psi_1 \,\mathcal{U}\, \psi_2) \in \text{closure}(\varphi)\}$

For a subformula ψ of φ let $\mathcal{A}_\varphi^\psi$ be the automaton A_φ with initial state ψ.
Claim: $\alpha \in \mathcal{L}(\mathcal{A}_\varphi^\psi) \;\Leftrightarrow\; \alpha \in \mathit{models}(\psi)$. Proof by structural induction. ■

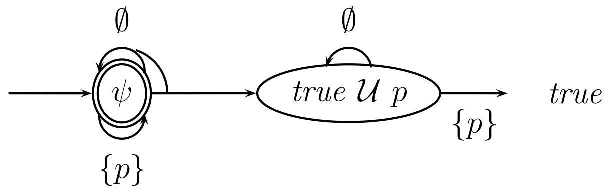
Example: $\varphi := \Diamond p \equiv (\mathit{true} \,\mathcal{U}\, p)$
$S = \{\mathit{true} \,\mathcal{U}\, p, \neg(\mathit{true} \,\mathcal{U}\, p), \mathit{true}, \neg\mathit{true}, p, \neg p\}$
$\delta(\mathit{true} \,\mathcal{U}\, p, \emptyset) = \delta(p,\emptyset) \vee (\delta(\mathit{true},\emptyset) \wedge \mathit{true} \,\mathcal{U}\, p) = \mathit{true} \,\mathcal{U}\, p$
$\delta(\mathit{true} \,\mathcal{U}\, p, \{p\}) = \delta(p,\{p\}) \vee (\delta(\mathit{true},\{p\}) \wedge \mathit{true} \,\mathcal{U}\, p) = \mathrm{T}$



$\varphi := \Box\Diamond p \equiv \neg(\mathit{true} \,\mathcal{U}\, \neg(\mathit{true} \,\mathcal{U}\, p))$

$$\begin{aligned}
\delta(\varphi,a) &= \overline{\delta(\neg(\mathit{true} \,\mathcal{U}\, p),a) \vee (\delta(\mathit{true},a) \wedge \mathit{true} \,\mathcal{U}\, \neg(\mathit{true} \,\mathcal{U}\, p))} \\
&= \delta(\mathit{true} \,\mathcal{U}\, p,a) \wedge \neg(\mathit{true} \,\mathcal{U}\, \neg(\mathit{true} \,\mathcal{U}\, p)) \\
&= (\delta(p,a) \vee (\delta(\mathit{true},a) \wedge \mathit{true} \,\mathcal{U}\, p)) \wedge \varphi \\
&= (\delta(p,a) \vee \mathit{true} \,\mathcal{U}\, p) \wedge \varphi
\end{aligned}$$

$\delta(\varphi,\emptyset) = \mathit{true} \,\mathcal{U}\, p \wedge \varphi$
$\delta(\varphi,\{p\}) = \varphi$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