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Automata, Games and Verification: Lecture 9

Definition 1 *Two nodes $x_1, x_2 \in T$ in a run tree (T, r) are* similar *if $|x_1| = |x_2|$ and $r(x_1) = r(x_2)$.*

Definition 2 *A run tree (T, r) is* memoryless *if for all similar nodes x_1 and x_2 and for all $y \in D^*$ we have that ($x_1 \cdot y \in T$ iff $x_2 \cdot y \in T$) and $r(x_1 \cdot y) = r(x_2 \cdot y)$.*

Theorem 1 *If an alternating Büchi Automaton $\mathcal{A}$ accepts a word α, then there exists a memoryless accepting run of $\mathcal{A}$ on α.*

Proof:

- Let (T, r) be an accepting run tree on α with directions D.
- We define $\gamma : T \to \omega$ (measures the number of steps since the last visit to F):
  - $\gamma(\epsilon) = 0$
  - $\gamma(x \cdot d) = \begin{cases} \gamma(x) + 1 & \text{if } x \notin F; \\ 0 & \text{otherwise;} \end{cases}$
- We define $\Delta : S \times \omega \to T$:
  $\Delta(s, n) =$ leftmost $y \in T$ with $|y| = n, r(y) = s$ and $(\forall z \in T, |z| = n \wedge r(z) = s \Rightarrow \gamma(z) \leq \gamma(y))$.
- We define (T', r'):
  - $\epsilon \in T$, $r'(\epsilon) = r(\epsilon)$;
  - for $n \in T'$, $d \in D$,
    $x \cdot d \in T'$ iff $\Delta(r'(n), |n|) \cdot d \in T$;
    $r'(n \cdot d) = r(\Delta(r'(n), |n|) \cdot d)$

Claim 1: (T', r') is a run of $\mathcal{A}$ on α.

- $r'(\epsilon) = r(\epsilon) = s_0$
- For $n \in T'$, let $q_n = \Delta(r'(n), |n|)$.
- For every $n \in T'$, $\{r(q_n \cdot d) \mid d \in D, q_n \cdot d \in T\} \models \delta(r(q_n), \alpha(|q_n|))$
  and therefore $\{r'(n \cdot d) \mid d \in D, n \cdot d \in T'\} \models \delta(r'(n), \alpha(|n|))$.

Claim 2: If (T, r) is accepting, then so is (T', r'). Proof by contradiction:

- Suppose (T', r') is not accepting, then there is an infinite branch $\pi : n_0, n_1, n_2, \ldots \in T'$ and $\exists k \in \omega$ such that $\forall j \geq k : r'(b_j) \notin F$.
- Let $m_i = \Delta(r'(n_i), |n_i|)$ for $i \geq k$.
- **Claim 2.1** : For every $m \in T'$, $\gamma(m) \leq \gamma(\Delta(r'(m), |m|))$. Proof by induction on the length of m:

  - for $m = \epsilon$, $\gamma(m) = 0$
  - for $m = m' \cdot d$ (where $d \in D$),
    * if $r(m') \in F$, then $\gamma(m) = 0$
    * if $r(m') \notin F$, then

$$
\begin{array}{ll}
 & \gamma(\Delta(r'(m' \cdot d), |m' \cdot d|)) \\
\geq & \quad (\Delta \text{ definition}) \\
 & \gamma(\Delta(r'(m'), |m'|) \cdot d) \\
= & \quad (\gamma \text{ definition}) \\
 & 1 + \gamma(\Delta(r'(m'), |m'|)) \\
\geq & \quad (\text{induction hypothesis}) \\
 & 1 + \gamma(m') \\
= & \quad (\gamma \text{ definition }) \\
 & \gamma(m' \cdot d)
\end{array}
$$

- We have,

$$
\begin{array}{ccccc}
\gamma(n_k) & < & \gamma(n_{k+1}) & < & \dots \\
/\!\!\wedge & & /\!\!\wedge & & \\
\gamma(m_k) & < & \gamma(m_{k+1}) & < & \dots
\end{array}
$$

  So, for any $k' > k, \gamma(m_k) \geq k' - k$.

  Since T is finitely branching, there must be a branch with an infinite suffix of non-F labeled positions. This contradicts our assumption that (T, r) is accepting.

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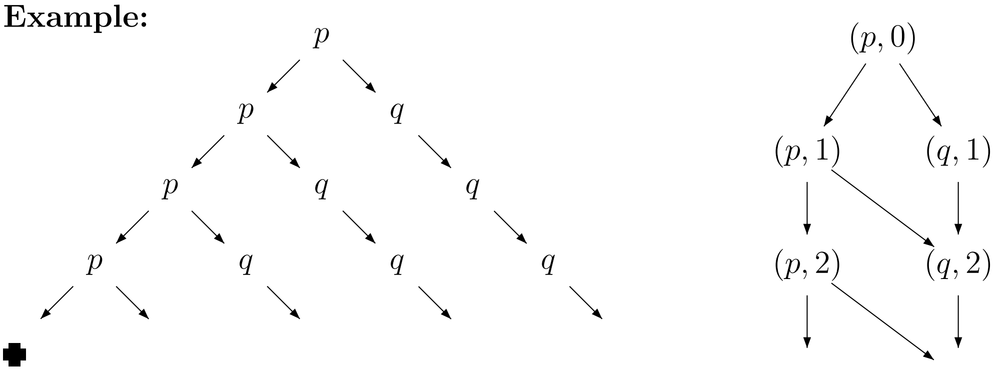
Definition 3 *A* run DAG *of an alternating Büchi Automaton $\mathcal{A}$ on word α is a DAG (V, E), where*

- $V \subseteq S \times \omega$
- $E \subseteq \bigcup_{i \in \omega} (S \times \{i\}) \times (S \times \{i+1\})$;
- $(s_0, 0) \in V$
- $\forall (s, i) \in V \ . \ \exists Y \subseteq S$ *s.t.*
  $Y \models \delta(s, \alpha(i)), Y \times \{i+1\} \subseteq V$ *and* $\{(s, i)\} \times (Y \times \{i+1\}) \subseteq E$.

Example:



Notation: Level $((V, E), i) = \{s \in S \mid (s, i) \in V\}$

Definition 4 *A run DAG is* accepting *if every path has infinitely many visits to $F \times \omega$.*

Corollary 1 *A word α is accepted by an alternating Büchi automaton $\mathcal{A}$ iff $\mathcal{A}$ has an accepting run DAG on α.*

Theorem 2 (Miyano and Hayashi, 1984) *For every alternating Büchi automaton $\mathcal{A}$, there exists a nondeterministic Büchi automaton $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

- $S' = 2^S \times 2^S$;
- $I' = \{(\{s_0\}, \emptyset)\}$;
- $F' = \{(X, \emptyset) \mid X \subseteq S\}$;
- $T' = \{((X, \emptyset), \sigma, (X', X' - F)) \mid X' \models \bigwedge_{s \in X} \delta(s, \sigma)\}$
  $\cup \{((x, W), \sigma, (X', W' \setminus F)) \mid W \neq \emptyset, W' \subseteq X', X' \models \bigwedge_{s \in X} \delta(s, \sigma),$
  $W' \models \bigwedge_{s \in W} \delta(s, \sigma)\}$.

$\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$**:**

- Let $\alpha \in L(\mathcal{A}')$ with accepting run

  $$r' : (X_0, W_0)(X_1, W_1)(X_2, W_2)\ldots$$

  where $W_0 = \emptyset, X_0 = \{s_0\}$.
- We construct the run DAG (V, E) for $\mathcal{A}$ on α:
  - $V = \bigcup_{i \in \omega} X_i \times \{i\}$;
  - $E = \bigcup_{i \in \omega} (\bigcup_{x \in X_i \setminus W_i} \{(x, i)\} \times (X_{i+1} \times \{i + 1\})$
    $\cup \bigcup_{x \in W_i} \{(x, i)\} \times \{(X_{i+1} \cap (F \cup W_{i+1})) \times \{i + 1\})$.
- (V, E) is an accepting run DAG:
  - $(s_0, 0) \in V$;
  - for $(x, i) \in V$:
    - if $x \in X_i \setminus W_i$, $X_{i+1} \models \delta(x, \alpha(i))$;
    - if $x \in W_i$, $X_{i+1} \cap (F \cup W_{i+1}) \models \delta(x, \alpha(i))$.

  - Every path through the run DAG visits F infinitely often (otherwise $W_i = \emptyset$ only for finitely many i).

$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$:

- Let $\alpha \in L(A')$ and (V, E) an accepting run DAG of $\mathcal{A}'$ on α.
- We construct a run

$$r' : (X_0, W_0)(X_1, W_1)(X_2, W_2)\dots$$

  on $\mathcal{A}$ as follows:

  - $X_0 = \{s_0\}, W_0 = \emptyset$;
  - for $i > 0$, $X_i = \mathit{Level}((V, E), i)$
    * if $W_i = \emptyset$ then $W_{i+1} = X_{i+1} \setminus F$,
    * otherwise,
      $W_{i+1} := \{y' \in S \setminus F \mid \exists (y, i) \in V, ((y, i), (y', i+1)) \in E, y \in W_i\}$.
- r' is an accepting run:
  - starts with $(\{s_0\}, \emptyset)$
  - obeys T':
    * for $x \in X_i \setminus W_i$, $X_{i+1} \models \delta(x, \alpha(i))$;
    * for $x \in W_i$, $X_{i+1} \cap (F \cup W_{i+1}) \models \delta(x, \alpha(i))$.
  - r' is accepting (otherwise there exists a path in (V, E) that is not accepting).

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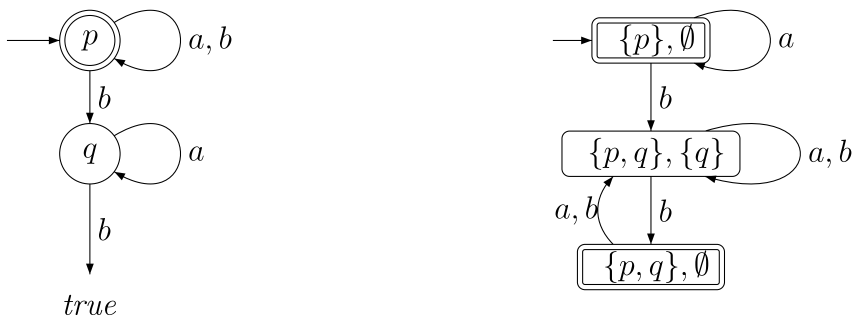
Example: We translate the following *universal* automaton (all branchings are conjunctions) into an equivalent nondeterministic automaton:



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Corollary 2 *A language is ω-regular iff it is recognizable by an alternating Büchi automaton.*

Proof:

Translation from nondeterministic Büchi automaton $(S, \{s_0\}, T, F)$ to alternating Büchi automaton (S, s_0, δ, F) with

- $\delta(s,\sigma) = \bigvee_{s' \in pr_3(T \cap \{s\} \times \{\sigma\} \times S)} s'$ for all $s \in S$

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Corollary 3 Satisfiability *of an LTL formula φ can be checked in time exponential in the length of φ.*

Corollary 4 Validity *of an LTL formula φ can be checked in time exponential in the length of φ.*

Comment: Acceptance of a word α by an alternating Büchi automaton can also be characterized by a game:

- Positions of player Blue: $B = S \times \omega$;
- Positions of player Green: $G = 2^S \times \omega$;
- Edges: $\{((s,i),(X,i)) \mid X \models \delta(s,\alpha(i))\}$
  $\cup \{((X,i),(s,i+1)) \mid s \in X\}$

Blue wins a play iff $F \times \omega$ is visited infinitely often.

The word α is accepted iff Blue has a strategy to win the game from position $(s_0, 0)$. **End Comment**