

Automata, Games & Verification

Summary #1

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- The **set of natural numbers** $\{0, 1, 2, 3, \dots\}$ is denoted by ω .
 - An **alphabet** Σ is a finite set of symbols.
 - An **infinite sequence/string/word** is a function from natural numbers to an alphabet:

$$\alpha : \omega \rightarrow \Sigma$$

$$\text{Notation: } \alpha = \alpha(0)\alpha(1)\alpha(2)\dots$$

- The **set of infinite words over alphabet** Σ is denoted Σ^ω .
- An **ω -language** L is a subset of Σ^ω .

Definition 1. A *(nondeterministic) Büchi automaton* \mathcal{A} over alphabet Σ is a tuple (S, I, T, F) :

- S : a finite set of *states*;
- $I \subseteq S$: a subset of *initial states*;
- $T \subseteq S \times \Sigma \times S$: a set of *transitions*;
- $F \subseteq S$: a subset of *accepting states*.

Definition 2. A *run* of a nondeterministic Büchi automaton \mathcal{A} on an infinite input word $\alpha = \sigma_0\sigma_1\sigma_2\dots$ is an infinite sequence of states s_0, s_1, s_2, \dots such that $s_0 \in I$ and for all $i \in \omega$, $(s_i, \sigma_i, s_{i+1}) \in T$.

Definition 3. A Büchi automaton \mathcal{A} is *deterministic* when

- I is a singleton and
- $\forall \sigma \in \Sigma, \forall s, s_0, s_1 \in S$.
 $(s, \sigma, s_0) \in T$ and $(s, \sigma, s_1) \in T \Rightarrow s_0 = s_1$.

Definition 4. The *infinity set of an infinite word* $\alpha \in \Sigma^\omega$ is defined as follows

$$In(\alpha) = \{\sigma \in \Sigma \mid \forall i \exists j . j \geq i \text{ and } \alpha(j) = \sigma\}.$$

Definition 5. [Büchi Acceptance Condition] A run $r = s_0s_1s_2\dots$ of a Büchi automaton \mathcal{A} is *accepting* if

$$In(r) \cap F \neq \emptyset.$$

Definition 6. A Büchi automaton \mathcal{A} *accepts* an infinite word α if there is an accepting run of \mathcal{A} on α .

Definition 7. The *language recognized by Büchi automaton \mathcal{A}* is defined as follows:

$$\mathcal{L}(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha\}.$$

Definition 8. An ω -language L is *Büchi recognizable* if there is a Büchi automaton \mathcal{A} such that $\mathcal{L}(\mathcal{A}) = L$.

Definition 9. A Büchi automaton is *complete* if

$$\forall s \in S, \sigma \in \Sigma, \exists s' \in S. (s, \sigma, s') \in T.$$

Theorem 1.

For every Büchi automaton \mathcal{A} , there is a complete Büchi automaton \mathcal{A}' such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

BACKGROUND: The Kleene Theorem

Definition 10. The *regular expressions* are defined as follows:

- The constants ϵ and \emptyset are regular expressions.
 $\mathcal{L}(\epsilon) = \{\epsilon\}, \mathcal{L}(\emptyset) = \emptyset.$
- If $a \in \Sigma$ is a symbol, then \mathbf{a} is a regular expression.
 $\mathcal{L}(\mathbf{a}) = \{a\}.$
- If E and F are regular expressions, then $E + F$ is a regular expression: $\mathcal{L}(E + F) = \mathcal{L}(E) \cup \mathcal{L}(F).$
- If E and F are regular expressions, then $E \cdot F$ is a regular expression: $\mathcal{L}(E \cdot F) = \{uv \mid u \in \mathcal{L}(E), v \in \mathcal{L}(F)\}.$
- If E is a regular expression, then E^* is a regular expression.
 $\mathcal{L}(E^*) = \{u_1 u_2 \dots u_n \mid n \in \omega, u_i \in \mathcal{L}(E) \forall 0 \leq i \leq n\}.$

Definition 11. *A language is **regular** if it is defined by a regular expression.*

Theorem 2. The Kleene Theorem

A language is regular iff it is recognized by some finite word automaton.