Automata, Games & Verification

Summary #2

Büchi's Characterization Theorem

Definition 1. *The ω-regular expressions are defined as follows.*

- *If R is an regular expression where $\epsilon \notin \mathcal{L}(R)$,*
  then R^ω is an ω-regular expression.
  $\mathcal{L}(R^\omega) = \mathcal{L}(R)^\omega$
  where $L^\omega = \{u_0 u_1 \ldots \mid u_i \in L, |u_i| > 0$ for all $i \in \omega\}$ for $L \subseteq \Sigma^*$.

- *If R is a regular expression and U is an ω-regular expression,*
  then $R \cdot U$ is an ω-regular expression.
  $\mathcal{L}(R \cdot U) = \mathcal{L}(R) \cdot \mathcal{L}(U)$
  where $L_1 \cdot L_2 = \{r \cdot u \mid r \in L_1, u \in L_2\}$ for $L_1 \subseteq \Sigma^*, L_2 \subseteq \Sigma^\omega$.

- *If U_1 and U_2 are ω-regular expressions,*
  then $U_1 + U_2$ is an ω-regular expression.
  $\mathcal{L}(U_1 + U_2) = \mathcal{L}(U_1) \cup \mathcal{L}(U_2)$.

Definition 2. *An ω-regular language is a finite union of ω-languages of the form $U \cdot V^\omega$ where $U, V \subseteq \Sigma^*$ are regular languages.*

Theorem 1. *If L_1 and L_2 are Büchi recognizable, then so is $L_1 \cup L_2$.*

Theorem 2. *If L_1 and L_2 are Büchi recognizable, then so is $L_1 \cap L_2$.*

Theorem 3. *If L_1 is a regular language and L_2 is Büchi recognizable, then $L_1 \cdot L_2$ is Büchi-recognizable.*

Theorem 4. *If L is a regular language then L^ω is Büchi recognizable.*

Theorem 5. [Büchi's Characterization Theorem (1962)] *An ω-language is Büchi recognizable iff it is ω-regular.*

Deterministic Büchi Automata

Theorem 6. *The language* $L = \{\alpha \in \Sigma^\omega \mid In(\alpha) = \{b\}\}$ *over* $\Sigma = \{a, b\}$ *is not recognizable by a deterministic Büchi automaton.*

Definition 3. [Substrings] *Let* $\alpha \in \Sigma^*$*. For two integers* $n \leq m$ *we define*

$$\alpha(n, m) = \alpha(n)\alpha(n+1)\ldots\alpha(m)\ .$$

Definition 4. [Limit] *For* $W \subseteq \Sigma^*$*:*

$$\overrightarrow{W} = \{\alpha \in \Sigma^\omega \mid \textit{there exist infinitely many } n \in \omega \textit{ s.t. } \alpha(0, n) \in W\}\ .$$

Theorem 7. *An* ω*-language* $L \subseteq \Sigma^\omega$ *is recognizable by a deterministic Büchi automaton iff there is a regular language* $W \subseteq \Sigma^*$ *s.t.* $L = \overrightarrow{W}$*.*