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Automata, Games and Verification: Lecture 10

13 Games

Definition 1 *A* game arena *is a triple* $\mathcal{A} = (V_0, V_1, E)$, *where*

- V_0 *and* V_1 *are disjoint sets of positions, called the positions of player* 0 *and* 1,
- $E \subseteq V \times V$ *for set* $V = V_0 \uplus V_1$ *of game positions,*
- *every position* $p \in V$ *has at least one outgoing edge* $(p, p') \in E$.

Definition 2 *A* play *is an infinite sequence* $\pi = p_0 p_1 p_2 \ldots \in V^\omega$ *such that* $\forall i \in \omega \,.\, (p_i, p_{i+1}) \in E$.

Definition 3 *A* strategy *for player* σ *is a function* $f_\sigma : V^* \cdot V_\sigma \to V$ *s.t.* $(p, p') \in E$ *whenever* $f(u \cdot p) = p'$.

Definition 4 *A play* $\pi = p_0, p_1, \ldots$ conforms to *strategy* f_σ *of player* σ *if* $\forall i \in \omega$. *if* $p_i \in V_\sigma$ *then* $p_{i+1} = f_\sigma(p_0, \ldots, p_i)$.

Definition 5

- *A* reachability game $\mathcal{G} = (\mathcal{A}, R)$ *consists of a game arena and a winning set of positions* $R \subseteq V$. *Player 0 wins a play* $\pi = p_0 p_1 \ldots$ *if* $p_i \in R$ *for some* $i \in \omega$, *otherwise Player 1 wins.*
- *A* Büchi game $\mathcal{G} = (\mathcal{A}, F)$ *consists of an arena* $\mathcal{A}$ *and a set* $F \subseteq V$. *Player 0 wins a play* π *if* $In(\pi) \cap F \neq \emptyset$, *otherwise Player 1 wins.*
- *A* Parity game $\mathcal{G} = (\mathcal{A}, c)$ *consists of an arena* $\mathcal{A}$ *and a coloring function* $c : V \to \mathbb{N}$. *Player 0 wins play* π *if* $\max\{c(q) \mid q \in In(\pi)\}$ *is even, otherwise Player 1 wins.*
- ...

Definition 6

- *A strategy* f_σ *is* p-winning *for player* σ *and position* p *if all plays that conform to* f_σ *and that start in* p *are won by Player* σ.
- *The* winning region *for player* σ *is the set of positions*

$$W_\sigma = \{p \in V \mid \textit{there is a strategy } f_\sigma \textit{ s.t. } f_\sigma \textit{ is } p\textit{-winning}\}.$$

Definition 7 *A game is* determined *if* $V = W_0 \cup W_1$.

Definition 8

- *A* memoryless *strategy for player* σ *is a function* $f_\sigma : V_\sigma \to V$ *which defines a strategy* $f'_\sigma(u \cdot v) = f_\sigma(v)$.
- *A game is* memoryless determined *if for every position some player wins the game with memoryless strategy.*

14 Solving Reachability Games

Attractor Construction:

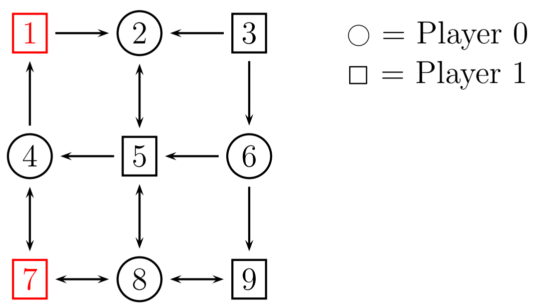
$$
\begin{aligned}
&Attr_\sigma^0(X) = \emptyset; \\
&Attr_\sigma^{i+1}(X) = Attr_\sigma^i(X) \\
&\qquad \cup \; \{p \in V_\sigma \mid \exists p' \,.\, (p,p') \in E \wedge p' \in Attr_\sigma^i(X) \cup X\} \\
&\qquad \cup \; \{p \in V_{1-\sigma} \mid \forall p' \,.\, (p,p') \in E \Rightarrow p' \in Attr_\sigma^i(X) \cup X\};
\end{aligned}
$$

$$Attr_\sigma^+(X) = \bigcup_{i\in\omega} Attr_\sigma^i(X).$$

$$Attr_\sigma(X) = Attr_\sigma^+(X) \cup X$$

The attractor construction solves the reachability game:
$W_0 = Attr_0(R)$, $W_1 = V \setminus W_0$.

Example: Consider the following reachability game with $R = \{1, 7\}$:



$$
\begin{aligned}
&Attr_0^0(\{1,7\}) = \emptyset; \\
&Attr_0^1(\{1,7\}) = \{4,8\}; \\
&Attr_0^2(\{1,7\}) = \{4,8,7,9\}; \\
&Attr_0^3(\{1,7\}) = \{4,6,7,8,9\}; \\
&Attr_0^4(\{1,7\}) = \{4,6,7,8,9\}; \\
&Attr_0^+(\{1,7\}) = \{4,6,7,8,9\}; \\
&Attr_0(\{1,7\}) = \{1,4,6,7,8,9\}.
\end{aligned}
$$

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Theorem 1 *Reachability games are memoryless determined.*

Proof:

Let $q \in V$.

1. If $p \in Attr_0(R)$, then $p \in W_0$, with memoryless strategy f_0:
   - Fix an arbitrary total ordering on V.
   - for $p \in V_0$ we define $f_0(q)$:

  - if $p \in Attr_0^i(R)$ for some smallest $i > 0$, choose the minimal $p' \in Attr_0^{i-1}(R) \cup R$.
  - otherwise, choose the minimal $p' \in V$ such that $(p, p') \in E$.
- Hence, if $p \in Attr_0^i(R)$ for some i, then any play that conforms to f_0 reaches R in at most i steps.

2. If $p \notin Attr_0(R)$, then $p \in W_1$ with memoryless strategy f_1:
   - for $p \in V_1$ we define $f_1(q)$:
     - if $p \in V_1 \setminus Attr_0(R)$, pick minimal $p' \in V \setminus Attr_0(R)$ such that $(p, p') \in E$. Such a p' must exist, since otherwise $p \in Attr_0(R)$.
     - otherwise, pick minimal $p' \in V$ such that $(p, p') \in E$.
   - Hence, if $p \in V \setminus Attr_0(R)$, then any play that conforms to f_1 never visits $Attr_0(R)$ and hence never R.

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15 Solving Büchi Games

Recurrence Construction:

$Recur_\sigma^0 = F;$
$Recur_\sigma^{i+1} = F \cap Attr_\sigma^+(Recur_\sigma^i);$

$Recur_\sigma = \bigcap_{i \in \omega} Recur_\sigma^i.$

The recurrence construction solves the Büchi game:
$W_0 = Attr_0(Recur_0)$, $W_1 = V \setminus W_0$.

Example: Same example as before, now as Büchi game with $F = \{1, 7\}$:

$Recur_0^0(\mathcal{G}) = \{1, 7\}$	$W_0 = \{4, 6, 7, 8, 9\}$
$Attr_0^+(\{1, 7\}, \mathcal{G}) = \{4, 6, 7, 8, 9\}$	$W_1 = \{1, 2, 3, 5\}$
$Recur_0^1(\mathcal{G}) = \{7\}$	
$Attr_0^+(\{7\}, \mathcal{G}) = \{4, 6, 7, 8, 9\}$	
$Recur_0(\mathcal{G}) = \{7\}$	
$Attr_0(\{7\}, \mathcal{G}) = \{4, 6, 7, 8, 9\}$	

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Theorem 2 *Büchi games are memoryless determined.*

Proof:

- If $p \in Attr_0(Recur_0)$, then $p \in W_0$, with memoryless strategy f_0:
  - Fix an arbitrary total ordering on V.
  - for $p \in V_0$ we define $f_0(q)$:
    - * if $p \in Attr_0(Recur_0)$, choose
      - · the minimal $p' \in Recur_0$, if $(p, p') \in E$ exists,
      - · the minimal $p' \in Attr_0^i(Recur_0)$ for minimal i such that $(p, p') \in E$ exists, otherwise.

  * if $p \notin Attr_0(Recur_0)$, choose minimal $p' \in V$ with $(p, p') \in E$.

- If $p \notin Attr_0(Recur_0)$, then $p \in W_1$ with memoryless strategy f_1: we define memoryless strategies f_1^i such that if a play starts in $p \in V \setminus Attr_0^+(Recur_0^i)$ and conforms to f_1^i, then there are at most i further visits to F (not counting a possible visit in the first position).
  - $f_1^0(p)$: choose minimal $p' \in V$ such that $(p, p') \in E$ and $p' \in V \setminus Attr_0(F)$.
  - if $p \in V \setminus Attr_0^+(Recur_0^i)$, $f_1^{i+1}(p) = f_1^i(p)$;
  - if $p \notin V \setminus Attr_0^+(Recur_0^i)$, i.e., if $p \in Attr_0^+(Recur_0^i) \setminus Attr_0^+(Recur_0^{i+1})$, then for $f_1^{i+1}(p)$ choose minimal p' such that $(p, p') \in E$ and $p' \in Attr_0^+(Recur_0^i) \setminus Attr_0^+(Recur_0^{i+1})$.
- Induction on i:
  - $i = 0$: Player 1 can avoid $Attr_0(F)$ and hence F;
  - $i + 1$:
    * case 1: play never reaches F;
    * case 2: play reaches $p' \in F \setminus Recur_0^{i+1} = F \setminus Attr_0^+(Recur_0^i) \subseteq V \setminus Attr_0^+(Recur_0^i)$; by induction hypothesis, at most i further visits to F, not counting the visit in p', hence a total of at most $i + 1$ visits from p.

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16 Parity Games

Assumptions:

- arena is finite or countably infinite.
- the number of colors is finite (max color k).

Theorem 3 *Parity games are memoryless determined.*

Proof:

Induction on k:

- $k = 0$: $W_0 = V, W_1 = \emptyset$. Memoryless winning strategy: fix arbitrary order on V. $f_0(p) = \min\{q \mid (p, q) \in E\}$.
- $k + 1$:
  - If $k + 1$ is even, consider player $\sigma = 0$, otherwise $\sigma = 1$.
  - Let $W_{1-\sigma}$ be the set of positions where Player $(1 - \sigma)$ has a memoryless winning strategy. We show that Player σ has a memoryless winning strategy from $V \setminus W_{1-\sigma}$.
  - Consider subgame $\mathcal{G}'$:
    * $V_0' = V_0 \setminus W_{1-\sigma}$;

    - $V'_1 = V_1 \setminus W_{1-\sigma}$;
    - $E' = E \cap (V' \times V')$;
    - $c'(p) = c(p)$ for all $p \in V'$.
- $\mathcal{G}'$ is still a game:
    - for $p \in V'_\sigma$, there is a $q \in V \setminus W_{1-\sigma}$ with $(p,q) \in E'$, otherwise $p \in W_{1-\sigma}$;
    - for $p \in V'_{1-\sigma}$, for all $q \in V$ with $(p,q) \in E$, $q \in V \setminus W_{1-\sigma}$, hence there is a $q \in V'$ with $(p,q) \in E$.
- Let $C'_i = \{p \in V' \mid c'(p) = i\}$.
- Let $Y = Attr'_\sigma(C'_{k+1})$. ($Attr'$: Attractor set on $\mathcal{G}'$)
- Let f_A be the attractor strategy on $\mathcal{G}'$ into C'_{k+1}.
- Consider subgame $\mathcal{G}''$:
    - $V''_0 = V'_0 \setminus Y$;
    - $V''_1 = V_1 \setminus Y$;
    - $E'' = E' \cap (V'' \times V'')$;
    - $c'' : V'' \to \{0, \dots, k\}; c''(p) = c'(p)$ for all $p \in V''$.
- $\mathcal{G}''$ is still a game.
- Induction hypothesis: $\mathcal{G}''$ is memoryless determined.
- Also: $W''_{1-\sigma} = \emptyset$ (because $W''_{1-\sigma} \subseteq W_{1-\sigma}$: assume Player $(1-\sigma)$ had a winning strategy from some position in V''. Then this strategy would win in $\mathcal{G}$, too, since Player σ has no chance to leave $\mathcal{G}''$ other than to $W_{1-\sigma}$.)
- Hence, there is a winning memoryless winning strategy f_{IH} for player σ from V''.
- We define:

$$f_\sigma(p) = \begin{cases} f_{IH}(p) & \text{if } p \in V''; \\ f_A(p) & \text{if } p \in Y \setminus C'_{k+1}; \\ \text{min. successor in } V \setminus W_{1-\sigma} & \text{if } p \in Y \cap C'_{k+1}; \\ \text{min. successor in } V & \text{otherwise.} \end{cases}$$

- f_σ is winning for Player σ on $V \setminus W_{1-\sigma}$.
  Consider a play that conforms to f_σ:
    - Case 1: Y is visited infinitely often.
      $\Rightarrow$ Player σ wins (inf. often even color $k+1$).
    - Case 2: Eventually only positions in V'' are visited.
      $\Rightarrow$ Since Player σ follows f_{IH}, Player σ wins.

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