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Automata, Games and Verification: Lecture 5

6 Muller Automata (Cont'd)

Theorem 1 *The languages recognizable by deterministic Muller automata are closed under Boolean operations (complementation, union, intersection).*

Proof:

- $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$:
  - $S' = S, I' = I, T' = T, \mathcal{F}' = 2^S \setminus \mathcal{F}$
- $\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$:
  - $S' = S_1 \times S_2, I' = I_1 \times I_2$,
  - $T' = \{((s_1, s_2), \sigma, (s_1', s_2')) \mid (s_1, \sigma, s_1') \in T_1, (s_2, \sigma, s_2') \in T_2\}$
  - $\mathcal{F}' = \{\{(p_1, q_1), \ldots, (p_n, q_n)\} \mid \{p_1, \ldots, p_n\} \in \mathcal{F}_1, \{q_1, \ldots, q_n\} \in \mathcal{F}_2\}$
- $\mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2) = \Sigma^\omega \setminus ((\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}_1)) \cap (\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}_2)))$.

■

Theorem 2 *A language $\mathcal{L}$ is recognizable by a deterministic Muller automaton iff $\mathcal{L}$ is a boolean combination of languages $\overrightarrow{W}$ where $W \subseteq \Sigma^*$ is regular.*

Proof:

(see Problem Set 4, Question 3) ■

7 McNaughton's Theorem

Theorem 3 (McNaughton's Theorem (1966)) *Every Büchi recognizable language is recognizable by a deterministic Muller automaton.*

Definition 1 *A Büchi automaton (S, I, T, F) is called* semi-deterministic *if $S = N \uplus D$ is a partition of S, $F \subseteq D$, $pr_3(T \cap (D \times \Sigma \times S)) \subseteq D$, and $(D, \{d\}, T \cap (D \times \Sigma \times D), F)$ is deterministic for every $d \in D$.*

Lemma 1 *For every Büchi automaton $\mathcal{A}$ there exists a semi-deterministic Büchi automaton $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

Given $\mathcal{A} = (S, I, T, F)$, we construct $\mathcal{A}' = (S', I', T', F')$:

- $S' = 2^S \uplus 2^S \times 2^S$;
- $I' = \{I\}$;
- $T' = \{(L, \sigma, L') \mid L' = pr_3(T \cap L \times \{\sigma\} \times S)\}$;
  $\cup\{(L, \sigma, (\{s'\}, \emptyset)) \mid \exists s \in L. (s, \sigma, s') \in T\}$
  $\cup\{((L_1, L_2), \sigma, (L_1', L_2')) \mid L_1 \neq L_2$
  $L_1' = pr_3(T \cap L_1 \times \{\sigma\} \times S)$,
  $L_2' = pr_3(T \cap L_1 \times \{\sigma\} \times F) \cup pr_3(T \cap L_2 \times \{\sigma\} \times S)\}$
  $\cup\{((L, L), \sigma, (L_1', L_2')) \mid L_1' = pr_3(T \cap L_1 \times \{\sigma\} \times S)$,
  $L_2' = pr_3(T \cap L_1 \times \{\sigma\} \times F)\}$
- $F' = \{(L, L) \mid L \neq \emptyset)\}$

$\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$:

- Let $\alpha \in \mathcal{L}(\mathcal{A}')$.
- Let $r' = P_0 P_1 \dots P_n (L_0, L_0')(L_1, L_1') \dots$ be an accepting run of $\mathcal{A}'$ on α.
- For $L_0 = \{s'\}$ there is a run prefix of $\mathcal{A}$ on $\alpha(0, n)$, $p_0 p_1 \dots p_n s'$ such that $p_j \in P_j$ and
- Let $i_0, i_1, \dots$ be an infinite sequence of indices such that $i_0 = 0$, $L_{i_j} = L_{i_j}'$, $L_{i_j} \neq \emptyset$ for all $j \in \omega$.
- For every $j > 1$, and every $s' \in L_{i_j}$ there exists a state $s \in L_{i_{j-1}}$ and a sequence $s = s_{i_{j-1}}, s_{i_{j-1}+1}, \dots, s_{i_j} = s'$ such that $(s_k, \alpha(k), s_{k+1}) \in T$ for all $k \in \{i_{j-1}, \dots, i_{i_j-1}\}$ and $s_k \in F$ for some $k \in \{i_{j-1}+1, \dots, i_{i_j}\}$.
  Let $predecessor(s', i_j) := s$,
  $run(s', i_0) = p_0 p_1 \dots p_n s'$ for $L_0 = \{s'\}$, and
  $run(s', i_j) = s_{i_{j-1}+1} s_{i_{j-1}+2} \dots s_{i_j}$, for $j > 0$.
- Consider the following $\left(\bigcup_{j\in\omega} L_{i_j} \times \{j\}\right)$-labeled tree:
  - the root is labeled with $(s, 0)$, where $L_0 = \{s\}$, and
  - the parent of each node labeled with (s', j) is labeled with $(predecessor(s', i_j), j - 1)$.
- The tree is infinite and finite-branching, and, hence, by König's Lemma, has an infinite branch $(s_{i_0}, i_0), (s_{i_1}, i_1), \dots$, corresponding to an accepting run of $\mathcal{A}$:

$$run(s_{i_0}, i_0) \cdot run(s_{i_1}, i_1) \cdot run(s_{i_2}, i_2) \cdot \dots$$

$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$:

- Let $\alpha \in \mathcal{L}(\mathcal{A})$.
- Let $r = s_0, s_1, \dots$ be an accepting run of $\mathcal{A}$ on α.
- Let i be an index s.t. $s_i \in F$ and for all $j \geq i$ there exists a $k > j$, such that

$$\{s \in S \mid s_i \rightarrow^{\alpha(i,k)} s\} = \{s \in S \mid s_j \rightarrow^{\alpha(j,k)} s\}.$$

  This index exists:

– "$\supseteq$" holds for all i, because there is a path through s_j.

– Assume that for all i, there is a $j \geq i$ s.t for all $k > j$ "$\supsetneq$" holds. Then there exists an i' s.t. $\{s \in S \mid s_{i'} \rightarrow^{\alpha(i',k)} s\} = \emptyset$ for all $k > i'$. Contradiction.

- We define a run r' of $\mathcal{A}'$:

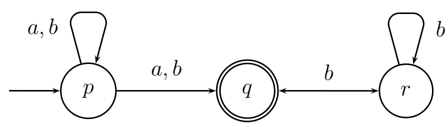
$$r' = P_0 \ldots P_{i-1}(\{s_i\}, \emptyset)(L_1, L'_1)(L_2, L'_2) \ldots$$

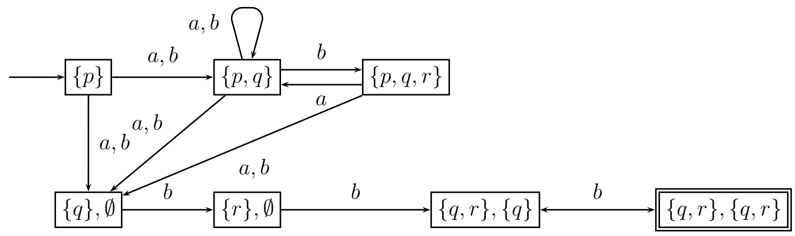
where $P_j = \{s \in S \mid p_0 \in I, p_0 \rightarrow^{\alpha(0,j)} s\}$, and L_j, L'_j are determined by the definition of $\mathcal{A}'$.

- We show that r' is accepting. Assume otherwise, and let m be an index such that $L_n \neq L'_n$ for all $n \geq m$.
- Then let $j > m$ be some index with $s_j \in F$; hence $s_j \in L'_j$. There exists a $k > j$ such that $L'_{k+1} = \{s \in S \mid s_j \rightarrow^{\alpha(j,k)} s\} = \{s \in S \mid s_i \rightarrow^{\alpha(i,k)} s\} = L_{k+1}$.
- Contradiction.

■

Example:





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