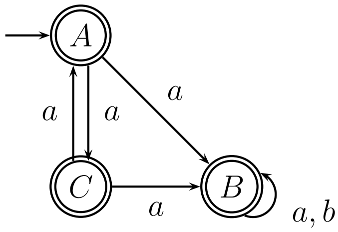
Bernd Finkbeiner Date: May 31, 2011

Automata, Games and Verification: Lecture 8

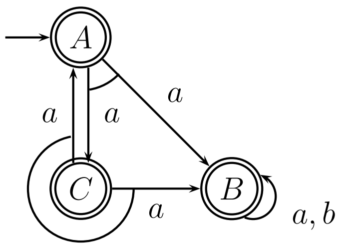
12 Alternating Automata

Example:

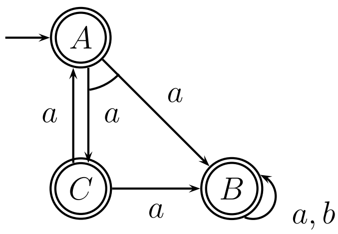
- Nondeterministic automaton, $L = a(a+b)^\omega$, disjunctive branching mode:



- universal automaton, $L = a^\omega$, conjunctive branching mode:



- Alternating automaton, both branching modes (arc between edges indicates universal branching mode), $L = aa(a+b)^\omega$



✚

Definition 1 *The* positive Boolean formulas *over a set X, denoted $\mathbb{B}^+(X)$, are the formulas built from elements of X, conjunction $\wedge$, disjunction $\vee$, true and false.*

Definition 2 *A set $Y \subseteq X$* satisfies *a formula $\varphi \in B^+(X)$, denoted $Y \models \varphi$, iff the truth assignment that assigns true to the members of Y and false to the members of $X \setminus Y$ satisfies φ.*

Definition 3 *An* alternating Büchi automaton *is a tuple* $\mathcal{A} = (S, s_0, \delta, F)$*, where:*

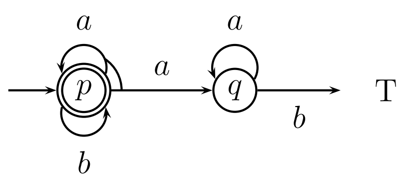
- *S is a finite set of states,*
- *$s_0 \in S$ is the initial state,*
- *$F \subseteq S$ is the set of accepting states, and*
- *$\delta : S \times \Sigma \to \mathbb{B}^+(S)$ is the transition function.*

A tree T over a set of *directions* D is a prefix-closed subset of D^*. The empty sequence ϵ is called the *root*. The children of a node $n \in T$ are the nodes children$(n) = \{n \cdot d \in T \mid d \in D\}$. A Σ-labeled tree is a pair (T, l), where $l : T \to \Sigma$ is the labeling function.

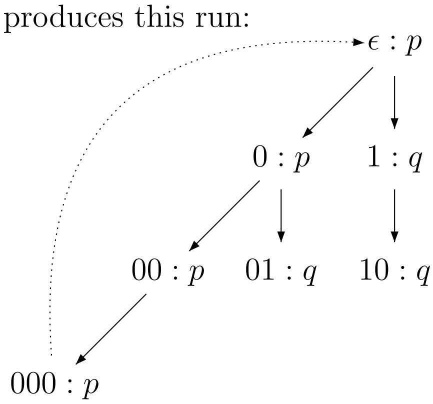
Definition 4 *A* run *of an alternating automaton on a word $\alpha \in \Sigma^\omega$ is an S-labeled* tree $\langle T, r\rangle$ *with the following properties:*

- *$r(\epsilon) = s_0$ and*
- *for all $n \in T$, if $r(n) = s$, then $\{r(n') \mid n' \in children(n)\}$ satisfies $\delta(s, \alpha(|n|))$.*

Example: $L = (\{a, b\}^* \, b)^\omega$



$S = \{p, q\}$
$F = \{p\}$
$\delta(p, a) = p \wedge q$
$\delta(p, b) = p$
$\delta(q, a) = q$
$\delta(q, b) = \mathrm{T}$
example word $w = (aab)^\omega$ produces this run:



(The dotted line means that the same tree would repeat there. Note that, in general, an alternating automaton may also have more than one run on a particular word—or no run at all.) ✚

Definition 5 *A* branch *of a tree T is a maximal sequence of words $n_0 n_1 n_2 \ldots$ such that $n_0 = \epsilon$ and n_{i+1} is a child of n_i for $i \geq 0$.*

Definition 6 *A run* (T, r) *is* accepting *iff, for every infinite branch* $n_0 n_1 n_2 \ldots$,

$$In(r(n_0)r(n_1)r(n_2)\ldots) \cap F \neq \emptyset.$$

Theorem 1 *For every LTL formula* φ*, there is an alternating Büchi automaton* $\mathcal{A}_\varphi$ *with* $\mathcal{L}(\mathcal{A}) = \mathit{models}(\varphi)$

Proof:

- $S = \text{closure}(\varphi) := \{\psi, \neg\psi \mid \psi \text{ is subformula of } \varphi\}$;
- $s_0 = \varphi$;
- $\delta(p, a) = \mathit{true}$ if $p \in a$, false if $p \notin a$;
  $\delta(\neg p, a) = \mathit{false}$ if $p \in a$, true if $p \notin a$;
  $\delta(\mathit{true}, a) = \mathit{true}$;
  $\delta(\mathit{false}, a) = \mathit{false}$;
- $\delta(\psi_1 \wedge \psi_2, a) = \delta(\psi_1, a) \wedge \delta(\psi_2, a)$;
- $\delta(\psi_1 \vee \psi_2, a) = \delta(\psi_1, a) \vee \delta(\psi_2, a)$;
- $\delta(\bigcirc \psi, a) = \psi$;
- $\delta(\psi_1 \,\mathcal{U}\, \psi_2, a) = \delta(\psi_2, a) \vee (\delta(\psi_1, a) \wedge \psi_1 \,\mathcal{U}\, \psi_2)$;
- $\delta(\neg\psi, a) = \overline{\delta(\psi, a)}$;
- $\overline{\psi} = \neg\psi$ for $\psi \in S$;
- $\overline{\neg\psi} = \psi$ for $\psi \in S$;
- $\overline{\alpha \wedge \beta} = \overline{\alpha} \vee \overline{\beta}$;
- $\overline{\alpha \vee \beta} = \overline{\alpha} \wedge \overline{\beta}$;
- $\overline{\mathit{true}} = \mathit{false}$;
- $\overline{\mathit{false}} = \mathit{true}$;
- $F = \{\neg(\psi_1 \,\mathcal{U}\, \psi_2) \in \text{closure}(\varphi)\}$

For a subformula ψ of φ let $\mathcal{A}_\varphi^\psi$ be the automaton A_φ with initial state ψ.
Claim: $\mathcal{L}(\mathcal{A}_\varphi^\psi) = \mathit{models}(\psi)$. Proof by structural induction. ■