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Problem Set 10
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Automata, Games, and Verification

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1. **Memoryless Strategies** (tutorial A: group G03, tutorial B: group G06)
   a) Give an example of a Muller game $\mathcal{G}$ with starting position q, for which neither Player 0 nor Player 1 has a winning memoryless strategy for plays beginning at q.
   b) Repeat part (a) for Streett games.
   c) Given a library of procedures for automata, describe a simple algorithm that verifies whether a given memoryless strategy is winning for Player 0 in a given Büchi game $\mathcal{G}$ beginning at some position q.

2. **Nondeterministic Strategies** (tutorial A: group G09, tutorial B: group G10)

   A *nondeterministic memoryless strategy* for Player 0 is a relation $R \subseteq (V_0 \times V) \cap E$. We say that Player 0 follows R in a play $p : p_0 p_1 p_2 \ldots$ if for all $i \in \omega$, $p_i \in V_0$ implies $(p_i, p_{i+1}) \in R$. The strategy R is winning for Player 0 if all plays played according to R are winning for Player 0.

   Prove or give a counterexample to the following statement:

   > If Player 0 has two winning nondeterministic memoryless strategies R_1 and R_2 for a Büchi game $\mathcal{G}$ from some position p, then $R_1 \cup R_2$ is a winning strategy for Player 0 in the game $\mathcal{G}$ from p.

3. **Update Networks** (tutorial A: group G11, tutorial B: group G12)

   In an *update game* $\mathcal{G} = (V_0, V_1, E)$, the players take turns, i.e, $E \subseteq (V_0 \times V_1) \cup (V_1 \times V_0)$. Player 0 wins a play if every position in $V = V_0 \cup V_1$ is visited infinitely often. An update game is an *update network* if Player 0 wins from every position. Update networks are for example of interest in the design of distributed networks (where each node needs to be updated with current information).

   We say that Player 1 is *forced* from a position $q \in V_1$ to move to a position $p \in V_0$, if (q, p) is the only edge in E from q. For $p \in V_0$, we define the following:

   $$\mathit{Forced}(p) = \{q \in V_1 \mid \text{ Player 1 is forced to move from } q \text{ to } p\}.$$

   A *forced cycle* is a sequence of positions

   $q_k, p_k, \ldots, q_2, p_2, q_1, p_1$

   such that $q_i \in \mathit{Forced}(p_i)$, $(p_{i+1}, q_i) \in E$ for all $1 \leq i \leq k$, and $(p_1, q_k) \in E$.

   Prove the following:
   a) If $\mathcal{G}$ is an update network, then for every position $p \in V_0$ there is a node $q \in V_1$ from which Player 1 is forced to move to p.
   b) If $\mathcal{G}$ is an update network for which $|V_0| > 1$, then for every $p \in V_0$, there exists a $p' \in V_0$ such that $p \neq p'$ and there is a node $q \in \mathit{Forced}(p)$ such that $(p', q), (q, p) \in E$.
   c) If $\mathcal{G}$ is an update network for which $|V_0| > 1$, then $\mathcal{G}$ has a forced cycle of length ≥ 4 (so $k \geq 2$).

4. **Mean Payoff Games** (Challenge problem)

A *mean payoff game* is a tuple (V_0, V_1, E, ν, d, w) where V_0, V_1, E denote Player 0's positions, Player 1's positions, and the edges, respectively. As usual, $V_0 \cap V_1 = \emptyset$ and $V := V_0 \cup V_1$. For each $p \in V$, there is a $p' \in V$ such that $(p, p') \in E$. ν and d are natural numbers, and $w : E \mapsto \{-d, \ldots, d\}$ assigns an integer value to each edge. Player 0 wins a play $v_0, v_1, \ldots$ iff

$$\liminf_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} w(v_{i-1}, v_i) \;\geq\; \nu.$$

For a given parity game, define a mean payoff game with the same game positions, such that winning memoryless strategies of the parity game are winning memoryless strategies of the mean payoff game and vice versa.