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Problem Set 2
Due: April 28, 2011

Automata, Games, and Verification

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1. ω-**Regular Expressions** (tutorial A: group G09, tutorial B: group G10)

   Represent each of the following ω-languages over the alphabet $\{a, b\}$ as a finite union of languages $V \cdot W^\omega$, where each V and W is recognizable by an automaton on finite words:

   a) $L_1 = \{\alpha \mid$ the maximal substrings of α consisting of only a's have even length$\}$

   b) $L_2 = \{\alpha \mid$ each a is preceeded by a b in $\alpha\}$

   c) $L_3 = \{\alpha \mid \alpha$ has no occurrence of $bab\}$

2. **Deterministic Büchi Automata** (tutorial A: group G11, tutorial B: group G12)

   Let Σ be an alphabet of the form $\Sigma = \Sigma_1 \times \Sigma_2 = \{(a, b) \mid a \in \Sigma_1, b \in \Sigma_2\}$, where Σ_1 and Σ_2 are also alphabets. Let L be a language over the alphabet Σ. We define the *projections* $pr_1(L)$ and $pr_2(L)$ as follows:

   $$pr_1(L) = \{\ u_0 u_1 u_2 \ldots \in \Sigma_1^\omega \mid \exists\ v_0 v_1 v_2 \ldots \in \Sigma_2^\omega \text{ s.t. } (u_0, v_0)(u_1, v_1)(u_2, v_2) \ldots \in L\ \}$$
   $$pr_2(L) = \{\ v_0 v_1 v_2 \ldots \in \Sigma_2^\omega \mid \exists\ u_0 u_1 u_2 \ldots \in \Sigma_1^\omega \text{ s.t. } (u_0, v_0)(u_1, v_1)(u_2, v_2) \ldots \in L\ \}$$

   Prove or give a counterexample to the following statements:

   a) Deterministic Büchi automata are closed under $\cap$.

   b) Deterministic Büchi automata are closed under $\cup$.

   c) Deterministic Büchi automata are closed under pr_1.

3. **Limit Operation** (tutorial A: group G13, tutorial B: group G14)

   For a finite word $\alpha \in \Sigma^*$ and two natural numbers $m, n \in \omega$ with $m < n$, $\alpha(m, n)$ denotes the substring from m to n: $\alpha(m, n) = \alpha(m)\,\alpha(m + 1)\ \ldots\ \alpha(n)$. For a finite-word language $W \subseteq \Sigma^*$, the *limit* of W is defined as $\overrightarrow{W} = \{\alpha \in \Sigma^\omega \mid$ there exist infinitely many $n \in \omega$ s.t. $\alpha(0, n) \in W\}$ (thus, an ω-string α belongs to $\overrightarrow{W}$ iff infinitely many prefixes of α belong to W).

   a) Let $V, W \subseteq \Sigma^*$ be two regular languages. Prove or give a counterexample to the following equation:

   $$\overrightarrow{(V \cdot W)} = V \cdot \overrightarrow{W}$$

   b) Let $\mathcal{A} = (S, \{s_0\}, T, F)$ be an automaton on finite words. Let $L_* = \mathcal{L}(\mathcal{A})$ be the language of $\mathcal{A}$ and let L_ω be the language of $\mathcal{A}$ when it is regarded as a Büchi automaton. Prove or give a counterexample for the following equation:

   $$L_\omega = \overrightarrow{L_*}$$

4. **Universal Projection** (tutorial A: group G15, tutorial B: group G02)

   We define the following "universal flavor" of projection (for $i \in \{1,2\}$ and $L \subseteq (\Sigma_1 \times \Sigma_2)^\omega$):

$$rp_i(L) = \{\alpha' \in \Sigma_i^\omega \mid \text{ for every } \alpha \in (\Sigma_1 \times \Sigma_2)^\omega \,.\, pr_i(\alpha) = \alpha' \Rightarrow \alpha \in L\}$$

   Show that if L is Büchi recognizable, then so is $rp_1(L)$.

   (For this question only, you may assume that the complement of any Büchi recognizable language is Büchi recognizable. We have not yet proved this, but this question is quite difficult without this theorem.)

5. **Projection and Büchi Recognizable Languages (Challenge)**

   (a) Prove that the projections $pr_1(L)$ and $pr_2(L)$ of a Büchi recognizable language L on the alphabet $\Sigma_1 \times \Sigma_2$ are Büchi recognizable.

   (b) Prove that the converse of (a) is false: Construct a non-Büchi recognizable ω-language L such that both $pr_1(L)$ and $pr_2(L)$ are Büchi recognizable.

   Hint: The language $L' = \{a^n b^n \mid n = 1, 2, 3, \ldots\}^\omega$ over the alphabet $\{a, b\}$ is not Büchi-recognizable. Some variation of L' is useful to construct the required language L.