Automata, Games & Verification

Summary #10

Today at 2:15pm in SR 016

Seminar "Games, Synthesis, and Robotics"

Non-communicative multi-robot coordination in dynamic environments

Games

Definition 1. *A game arena is a triple* $\mathcal{A} = (V_0, V_1, E)$, *where*

- V_0 *and* V_1 *are disjoint sets of positions, called the positions of player* 0 *and* 1,
- $E \subseteq V \times V$ *for set* $V = V_0 \uplus V_1$ *of game positions,*
- *every position* $p \in V$ *has at least one outgoing edge* $(p, p') \in E$.

Definition 2.

- *A reachability game* $\mathcal{G} = (\mathcal{A}, R)$ *consists of a game arena and a winning set of positions* $R \subseteq V$. *Player 0 wins a play* $\pi = p_0 p_1 \ldots$ *if* $p_i \in R$ *for some* $i \in \omega$, *otherwise Player 1 wins.*
- *A Büchi game* $\mathcal{G} = (\mathcal{A}, F)$ *consists of an arena* $\mathcal{A}$ *and a set* $F \subseteq V$. *Player 0 wins a play* π *if* $In(\pi) \cap F \neq \emptyset$, *otherwise Player 1 wins.*
- $\ldots$

Definition 3. *A play is an infinite sequence* $\pi = p_0 p_1 p_2 \ldots \in V^\omega$ *such that* $\forall i \in \omega \;.\; (p_i, p_{i+1}) \in E$.

Definition 4. *A strategy for player* σ *is a function* $f_\sigma : V^* \cdot V_\sigma \to V$ *s.t.* $(p, p') \in E$ *whenever* $f(u \cdot p) = p'$.

Definition 5. *A play* $\pi = p_0, p_1, \ldots$ *conforms to strategy* f_σ *of player* σ *if* $\forall i \in \omega \;.$ *if* $p_i \in V_\sigma$ *then* $p_{i+1} = f_\sigma(p_0, \ldots, p_i)$.

Definition 6.

- *A strategy* f_σ *is* p*-winning for player* σ *and position* p *if all plays that conform to* f_σ *and that start in* p *are won by Player* σ.
- *The winning region for player* σ *is the set of positions*

$$W_\sigma = \{p \in V \mid \text{there is a strategy } f_\sigma \text{ s.t. } f_\sigma \text{ is } p\text{-winning}\}.$$

Definition 7. *A game is determined if* $V = W_0 \cup W_1$.

Definition 8.

- *A memoryless strategy for player* σ *is a function* $f_\sigma : V_\sigma \to V$ *which defines a strategy* $f'_\sigma(u \cdot v) = f(v)$.

- *A game is memoryless determined if for every position some player wins the game with memoryless strategy.*

Solving Reachability Games

Attractor construction:

$$Attr_\sigma^0(X) = \emptyset;$$

$$\begin{aligned} Attr_\sigma^{i+1}(X) = {} & Attr_\sigma^i(X) \\ & \cup \ \{p \in V_\sigma \mid \exists p' \ . \ (p,p') \in E \wedge p' \in Attr_\sigma^i(X) \cup X\} \\ & \cup \ \{p \in V_{1-\sigma} \mid \forall p' \ . \ (p,p') \in E \Rightarrow p' \in Attr_\sigma^i(X) \cup X\}; \end{aligned}$$

$$Attr_\sigma^+(X) = \bigcup_{i \in \omega} Attr_\sigma^i(X).$$

$$Attr_\sigma(X) = Attr_\sigma^+(X) \cup X$$

Attractor strategy:

- Fix an arbitrary total ordering on V.

- for $p \in V_0$ we define $f_0(q)$:
  - if $p \in Attr_0^i(R)$ for some smallest $i > 0$, choose the minimal $p' \in Attr_0^{i-1}(R) \cup R$.
  - otherwise, choose the minimal $p' \in V$ such that $(p, p') \in E$.

Solving Büchi Games

Recurrence construction:

$$Recur_\sigma^0 = F;$$

$$Recur_\sigma^{i+1} = F \cap Attr_\sigma^+(Recur_\sigma^i);$$

$$Recur_\sigma = \bigcap_{i \in \omega} Recur_\sigma^i.$$

Theorem 1. *Reachability and Büchi games are memoryless determined.*

Theorem 2. *Parity games are memoryless determined.*

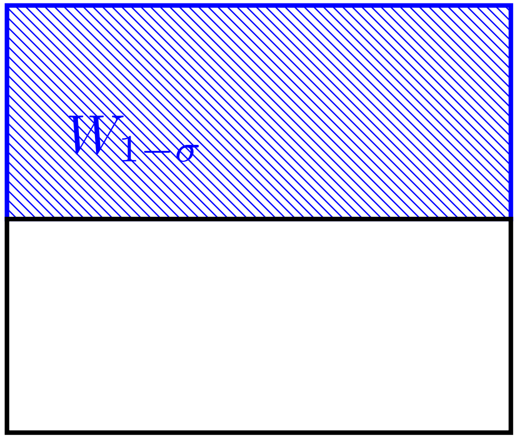
Assumptions:

- arena is finite or countably infinite.
- the number of colors is finite (max color k).

Proof by induction on k:

- $k = 0$: $W_0 = V, W_1 = \varnothing$. Memoryless winning strategy: fix arbitrary order on V. $f_0(p) = \min\{q \mid (p, q) \in E\}$.
- $k + 1$:
  - If $k + 1$, consider player $\sigma = 0$, otherwise $\sigma = 1$.
  - Let $W_{1-\sigma}$ be the set of positions where Player $(1-\sigma)$ has a memoryless winning strategy. We show that Player σ has a memoryless winning strategy from $V \setminus W_{1-\sigma}$.

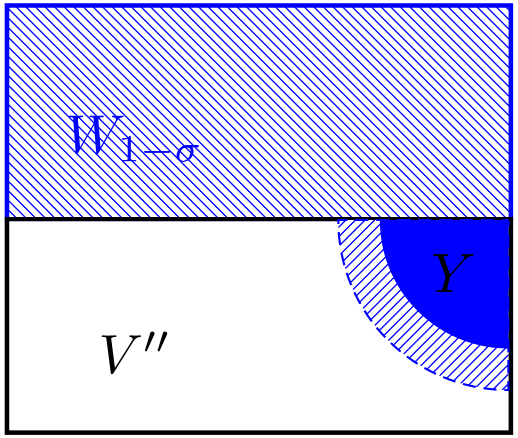
- Consider subgame $\mathcal{G}'$:
  * $V_0' = V_0 \setminus W_{1-\sigma}$;
  * $V_1' = V_1 \setminus W_{1-\sigma}$;
  * $E' = W \cap (V' \times V')$;
  * $c'(p) = c(p)$ for all $p \in V'$.



- $\mathcal{G}'$ is still a game.

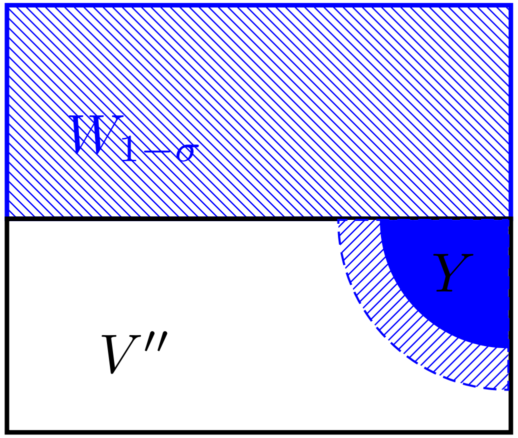
- Let $C_i' = \{p \in V' \mid c'(p) = i\}$.
- Let $Y = Attr'_\sigma(C'_{k+1})$. ($Attr'$: Attractor set on $\mathcal{G}'$)
- Let f_A be the attractor strategy on $\mathcal{G}'$ into C'_{k+1}.

- Consider subgame $\mathcal{G}''$:
  - $V_0'' = V_0' \setminus Y$;
  - $V_1'' = V_1 \setminus Y$;
  - $E' = W \cap (V'' \times V'')$;
  - $C'' : V'' \to \{0, \ldots, k\}; c''(p) = c'(p)$ for all $p \in V''$.



- $\mathcal{G}''$ is still a game.

- Induction hypothesis: $\mathcal{G}''$ is memoryless determined.
- Also: $W''_{1-\sigma} = \varnothing$ (because $W''_{1-\sigma} \subseteq W_{1-\sigma}$: assume Player $(1-\sigma)$ had a winning strategy from some position in V''. Then this strategy would win in $\mathcal{G}$, too, since Player σ has no chance to leave $\mathcal{G}''$ other than to $W_{1-\sigma}$.)
- Hence, there is a winning memoryless winning strategy f_{IH} for player σ from V''.



– We define:

$$f_\sigma(p) = \begin{cases} f_{IH}(p) & \text{if } p \in V''; \\ f_A(p) & \text{if } p \in Y \setminus C'_{k+1}; \\ \text{min. successor in } V \setminus W_{1-\sigma} & \text{if } p \in Y \cap C'_{k+1}; \\ \text{min. successor in } V & \text{otherwise.} \end{cases}$$

– f_σ is winning for Player σ on $V \setminus W_{1-\sigma}$.
Consider a play that conforms to f_σ:
  * Case 1: Y is visited infinitely often.
    ⇒ Player σ wins (inf. often even color $k+1$).
  * Case 2: Eventually only positions in V'' are visited.
    ⇒ Since Player σ follows f_{IH}, Player σ wins.