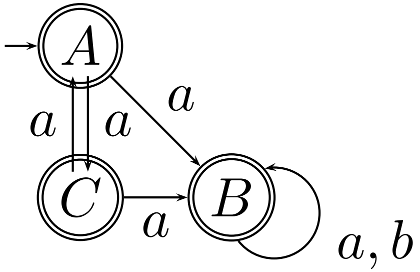
# Automata, Games & Verification

Summary #8

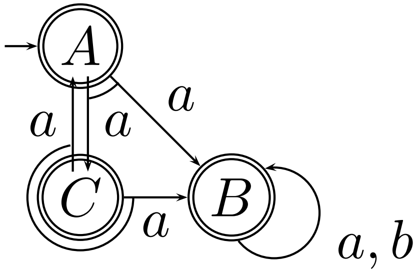
Today at 2:15pm in SR 016
**Seminar "Games, Synthesis, and Robotics"**
*Church's Problem Revisited*

# Alternating Automata

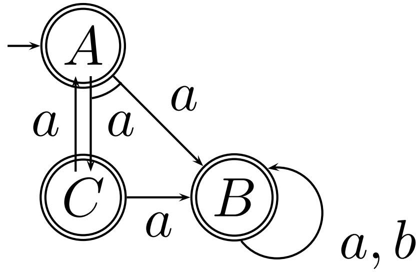
- nondeterministic automaton,
  $L = a(a+b)^\omega$:




- universal automaton, $L = a^\omega$:




- alternating automaton,
  $L = aa(a+b)^\omega$



**Definition 1.** *An alternating Büchi automaton is a tuple* $\mathcal{A} = (S, s_0, \delta, F)$, *where:*

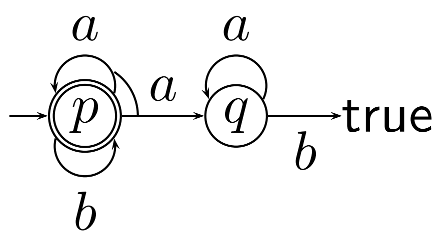
- $S$ *is a finite set of states,*
- $s_0 \in S$ *is the initial state,*
- $F \subseteq S$ *is the set of accepting states, and*
- $\delta : S \times \Sigma \rightarrow \mathbb{B}^+(S)$ *is the transition function.*

**Definition 2.** *A run of an alternating automaton on a word* $\alpha \in \Sigma^\omega$ *is an* $S$*-labeled tree* $\langle T, r \rangle$ *with the following properties:*

- $r(\epsilon) = s_0$ *and*
- *for all* $n \in T$,
  *if* $r(n) = s$, *then* $\{r(n') \mid n' \in$ *children*$(n)\}$ *satisfies* $\delta(s, \alpha(|n|))$.

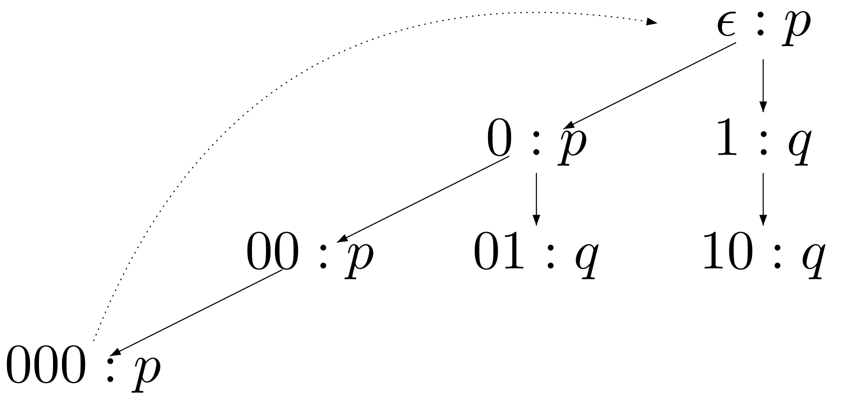
Example:

$L = (\{a, b\}^* \, b)^\omega$




$S = \{p, q\}$
$F = \{p\}\,;$
$\delta(p, a) = p \wedge q$; $\delta(p, b) = p$; $\delta(q, a) = q$; $\delta(q, b) =$ true

example word $w = (aab)^\omega$ has the following run:



**Theorem 1.** *For every LTL formula $\varphi$, there is an alternating Büchi automaton $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) = models(\varphi)$*

- $S = \text{closure}(\varphi) := \{\psi, \neg\psi \mid \psi \text{ is subformula of } \varphi\}$;

- $s_0 = \varphi$;

- $\delta(p, a) = \text{true if } p \in a, \text{ false if } p \notin a$;
  $\delta(\neg p, a) = \text{false if } p \in a, \text{ true if } p \notin a$;
  $\delta(\text{true}, a) = \text{true}$;
  $\delta(\text{false}, a) = \text{false}$;

- $\delta(\psi_1 \wedge \psi_2, a) = \delta(\psi_1, a) \wedge \delta(\psi_2, a)$;

- $\delta(\psi_1 \vee \psi_2, a) = \delta(\psi_1, a) \vee \delta(\psi_2, a)$;

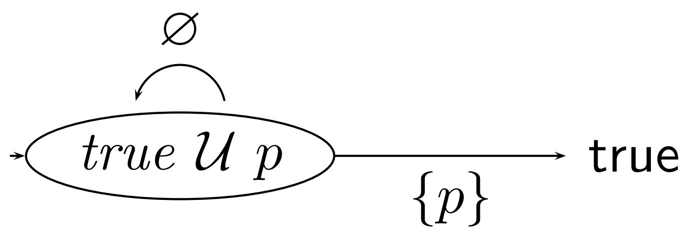
- $\delta(\bigcirc\, \psi, a) = \psi$;
- $\delta(\psi_1 \,\mathcal{U}\, \psi_2, a) = \delta(\psi_2, a) \vee (\delta(\psi_1, a) \wedge \psi_1 \,\mathcal{U}\, \psi_2)$;
- $\delta(\neg\psi, a) = \overline{\delta(\psi, a)}$;
- $\overline{\psi} = \neg\psi$ for $\psi \in S$;
- $\overline{\neg\psi} = \psi$ for $\psi \in S$;
- $\overline{\alpha \wedge \beta} = \overline{\alpha} \vee \overline{\beta}$;
- $\overline{\alpha \vee \beta} = \overline{\alpha} \wedge \overline{\beta}$;
- $\overline{\text{true}} = \text{false}$; $\overline{\text{false}} = \text{true}$;
- $F = \{\neg(\psi_1 \,\mathcal{U}\, \psi_2) \in \text{closure}(\varphi)\}$

## Example:

$\varphi := \Diamond p \equiv (\mathsf{true}\ \mathcal{U}\ p)$

$S = \{\mathsf{true}\ \mathcal{U}\ p, \neg(\mathsf{true}\ \mathcal{U}\ p), \mathsf{true}, \neg\mathsf{true}, p, \neg p\}$

$\delta(\mathsf{true}\ \mathcal{U}\ p, \varnothing) = \delta(p, \varnothing) \vee (\delta(\mathsf{true}, \varnothing) \wedge \mathsf{true}\ \mathcal{U}\ p) = \mathsf{true}\ \mathcal{U}\ p$
$\delta(\mathsf{true}\ \mathcal{U}\ p, \{p\}) = \delta(p, \{p\}) \vee (\delta(\mathsf{true}, \{p\}) \wedge \mathsf{true}\ \mathcal{U}\ p) = \mathsf{true}$



$\varphi := \Box\Diamond p \equiv \neg(\textsf{true}\ \mathcal{U}\ \neg(\textsf{true}\ \mathcal{U}\ p))$

$$\begin{aligned}
\delta(\varphi, a) &= \overline{\delta(\neg(\textsf{true}\ \mathcal{U}\ p), a) \vee (\delta(\textsf{true}, a) \wedge \textsf{true}\ \mathcal{U}\ \neg(\textsf{true}\ \mathcal{U}\ p))} \\
&= \delta(\textsf{true}\ \mathcal{U}\ p, a) \wedge \neg(\textsf{true}\ \mathcal{U}\ \neg(\textsf{true}\ \mathcal{U}\ p)) \\
&= (\delta(p, a) \vee (\delta(\textsf{true}, a) \wedge \textsf{true}\ \mathcal{U}\ p)) \wedge \varphi \\
&= (\delta(p, a) \vee \textsf{true}\ \mathcal{U}\ p) \wedge \varphi
\end{aligned}$$

$$\delta(\varphi, \varnothing) = \textsf{true}\ \mathcal{U}\ p \wedge \varphi$$

$$\delta(\varphi, \{p\}) = \varphi$$