Automata, Games & Verification

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Definition 1. *A (nondeterministic) Büchi automaton* $\mathcal{A}$ *over alphabet* Σ *is a tuple* (S, I, T, F)*:*

- S *: a finite set of states;*
- $I \subseteq S$ *: a subset of initial states;*
- $T \subseteq S \times \Sigma \times S$ *: a set of transitions;*
- $F \subseteq S$ *: a subset of accepting states.*

Definition 2. *A run of a nondeterministic Büchi automaton* $\mathcal{A}$ *on an infinite input word* $\alpha = \sigma_0\sigma_1\sigma_2\ldots$ *is an infinite sequence of states* $s_0, s_1, s_2, \ldots$ *such that* $s_0 \in I$ *and for all* $i \in \omega$, $(s_i, \sigma_i, s_{i+1}) \in T$.

Definition 3. *A Büchi automaton $\mathcal{A}$ is deterministic when*

- *I is a singleton and*
- $\forall \sigma \in \Sigma, \forall s, s_0, s_1 \in S$.
  $(s, \sigma, s_0) \in T$ *and* $(s, \sigma, s_1) \in T \;\Rightarrow\; s_0 = s_1$.

Definition 4. *The infinity set of an infinite word $\alpha \in \Upsilon^\omega$ is defined as follows*

$$In(\alpha) \;=\; \{ \upsilon \in \Upsilon \,|\, \forall i \exists j \,.\, j \geq i \text{ and } \alpha(j) = \upsilon \}.$$

Definition 5. [Büchi Acceptance Condition] *A run $r = s_0 s_1 s_2 \ldots$ of a Büchi automaton $\mathcal{A}$ is accepting if*

$$In(r) \cap F \neq \emptyset.$$

Definition 6. *A Büchi automaton $\mathcal{A}$ accepts an infinite word α if there is an accepting run of $\mathcal{A}$ on α.*

Definition 7. *The language recognized by Büchi automaton $\mathcal{A}$ is defined as follows:*

$$\mathcal{L}(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha\}.$$

Definition 8. *An ω-language L is Büchi recognizable if there is a Büchi automaton $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = L$.*

Definition 9. *A Büchi automaton is complete if*

$$\forall s \in S, \sigma \in \Sigma, \exists s' \in S \,.\, (s, \sigma, s') \in T.$$

Theorem 1. *For every Büchi automaton $\mathcal{A}$, there is a complete Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Büchi's Characterization Theorem

Definition 10. *The* ω-regular expressions *are defined as follows.*

- *If* R *is an regular expression where* $\varepsilon \notin \mathcal{L}(R)$*,*
  then R^ω *is an* ω*-regular expression.*
  $\mathcal{L}(R^\omega) = \mathcal{L}(R)^\omega$
  where $L^\omega = \{u_0 u_1 \ldots \mid u_i \in L, |u_i| > 0$ *for all* $i \in \omega\}$ *for* $L \subseteq \Sigma^*$.

- *If* R *is a regular expression and* U *is an* ω*-regular expression,*
  then $R \cdot U$ *is an* ω*-regular expression.*
  $\mathcal{L}(R \cdot U) = \mathcal{L}(R) \cdot \mathcal{L}(U)$
  where $L_1 \cdot L_2 = \{r \cdot u \mid r \in L_1, u \in L_2\}$ *for* $L_1 \subseteq \Sigma^*, L_2 \subseteq \Sigma^\omega$.

- *If* U_1 *and* U_2 *are* ω*-regular expressions,*
  then $U_1 + U_2$ *is an* ω*-regular expression.*
  $\mathcal{L}(U_1 + U_2) = \mathcal{L}(U_1) \cup \mathcal{L}(U_2)$.

Definition 11. *An* ω-regular *language is a finite union of* ω*-languages of the form* $U \cdot V^{\omega}$ *where* $U, V \subseteq \Sigma^*$ *are regular languages.*

Theorem 2. *If* L_1 *and* L_2 *are Büchi recognizable, then so is* $L_1 \cup L_2$.

Theorem 3. *If* L_1 *and* L_2 *are Büchi recognizable, then so is* $L_1 \cap L_2$.

Theorem 4. *If* L_1 *is a regular language and* L_2 *is Büchi recognizable, then* $L_1 \cdot L_2$ *is Büchi-recognizable.*

Theorem 5. *If* L *is a regular language then* L^{ω} *is Büchi recognizable.*

Theorem 6. [Büchi's Characterization Theorem (1962)] *An* ω*-language is Büchi recognizable iff it is* ω*-regular.*