Automata, Games & Verification

#4

Deterministic Büchi Automata

Theorem 1. *The* ω*-language* $(a+b)^* b^\omega$ *is not recognizable by a deterministic Büchi automaton.*

Definition 1. [Substrings] *Let* $\alpha \in \Sigma^*$*. For two integers* $n \leq m$ *we define*

$$\alpha(n,m) = \alpha(n)\alpha(n+1)\ldots\alpha(m)\,.$$

Definition 2. [Limit] *For* $W \subseteq \Sigma^*$*:*

$$\overrightarrow{W} = \{\alpha \in \Sigma^\omega \mid \text{there exist infinitely many } n \in \omega \text{ s.t. } \alpha(0,n) \in W\}\,.$$

Theorem 2. *An* ω*-language* $L \subseteq \Sigma^\omega$ *is recognizable by a deterministic Büchi automaton iff there is a regular language* $W \subseteq \Sigma^*$ *s.t.* $L = \overrightarrow{W}$*.*

Theorem 3. *For any deterministic Büchi automaton $\mathcal{A}$, there exists a Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.*

Proof: We construct $\mathcal{A}'$ as follows:

- $S' = (S \times \{0\}) \cup ((S \setminus F) \times \{1\})$.
- $I' = I \times \{0\}$.
- $T' = \{((s,0),\sigma,(s',0)) \mid (s,\sigma,s') \in T\}$
  $\cup \ \{((s,0),\sigma,(s',1)) \mid (s,\sigma,s') \in T, s' \in S - F\}$
  $\cup \ \{((s,1),\sigma,(s,1)) \mid (s,\sigma,s') \in T, s' \in S - F\}$.
- $F' = (S - F) \times \{1\}$.

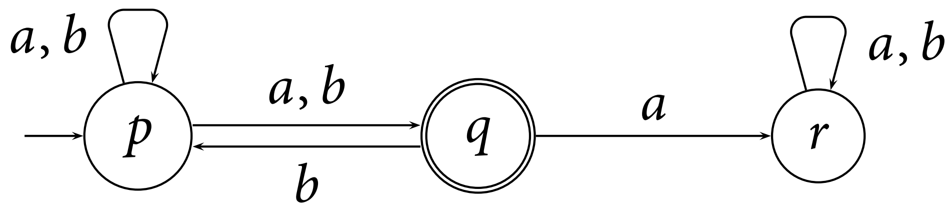
■

Complementation of Nondeterministic Büchi Automata

Definition 3. *Let $\mathcal{A} = (S, I, T, F)$ be a nondeterministic Büchi automaton. The run DAG of $\mathcal{A}$ on a word $\alpha \in \Sigma^\omega$ is the directed acyclic graph $G = (V, E)$ where*

- $V = \bigcup_{l \geq 0}(S_l \times \{l\})$ *where* $S_0 = I$ *and* $S_{l+1} = \bigcup_{s \in S_l, (s, \alpha(l), s') \in T}\{s'\}$
- $E = \{(\langle s, l\rangle, \langle s', l+1\rangle) \mid l \geq 0, (s, \alpha(l), s') \in T\}$

A path in a run DAG is accepting iff it visits F infinitely often.
The automaton accepts α if some path is accepting.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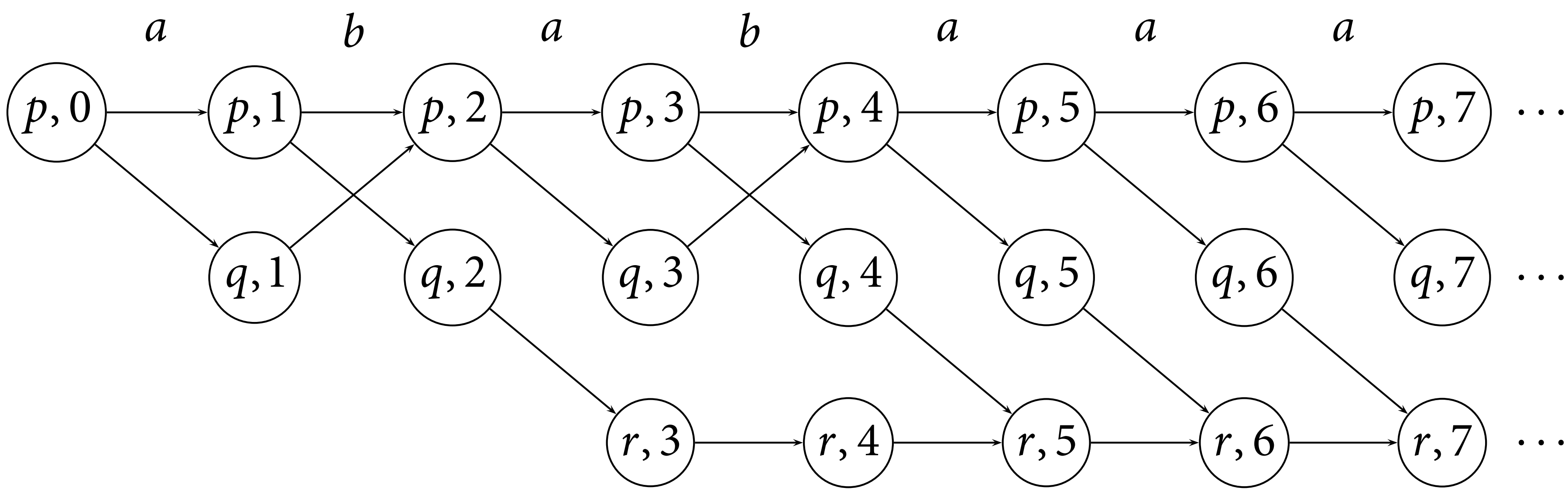
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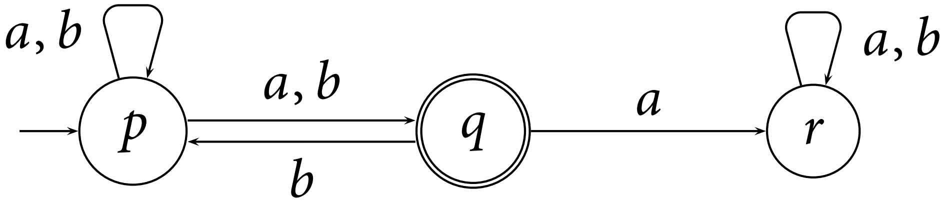
Definition 4. *A ranking for G is a function $f : V \to \{0, \ldots, 2 \cdot |S|\}$ such that*

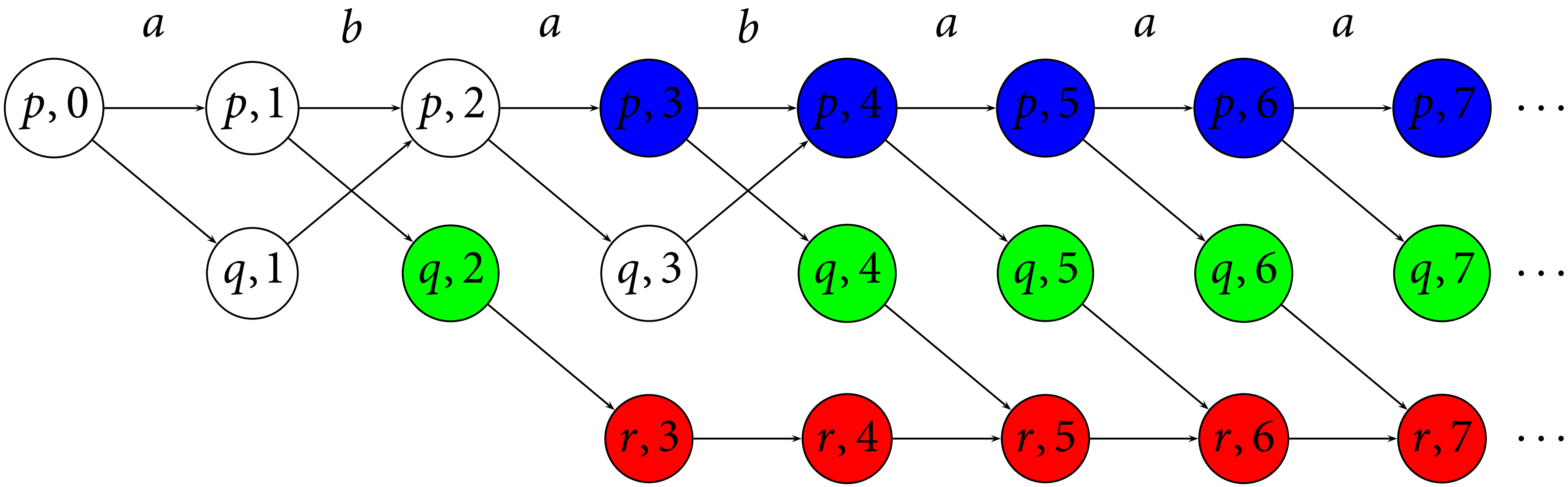
- *for all $\langle s, l \rangle \in V$, if $f(\langle s, l \rangle)$ is odd then $s \notin F$;*
- *for all $(\langle s, l \rangle, \langle s', l' \rangle) \in E$, $f(\langle s', l' \rangle) \leq f(\langle s, l \rangle)$.*

A ranking is odd iff for all paths $\langle s_0, l_0 \rangle, \langle s_1, l_1 \rangle, \langle s_2, l_2 \rangle, \ldots$ in G,
there is a $i \geq 0$ such that $f(\langle s_i, l_i \rangle)$ is odd and,
for all $j \geq 0$, $f(\langle s_{i+j}, l_{i+j} \rangle) = f(\langle s_i, l_i \rangle)$.

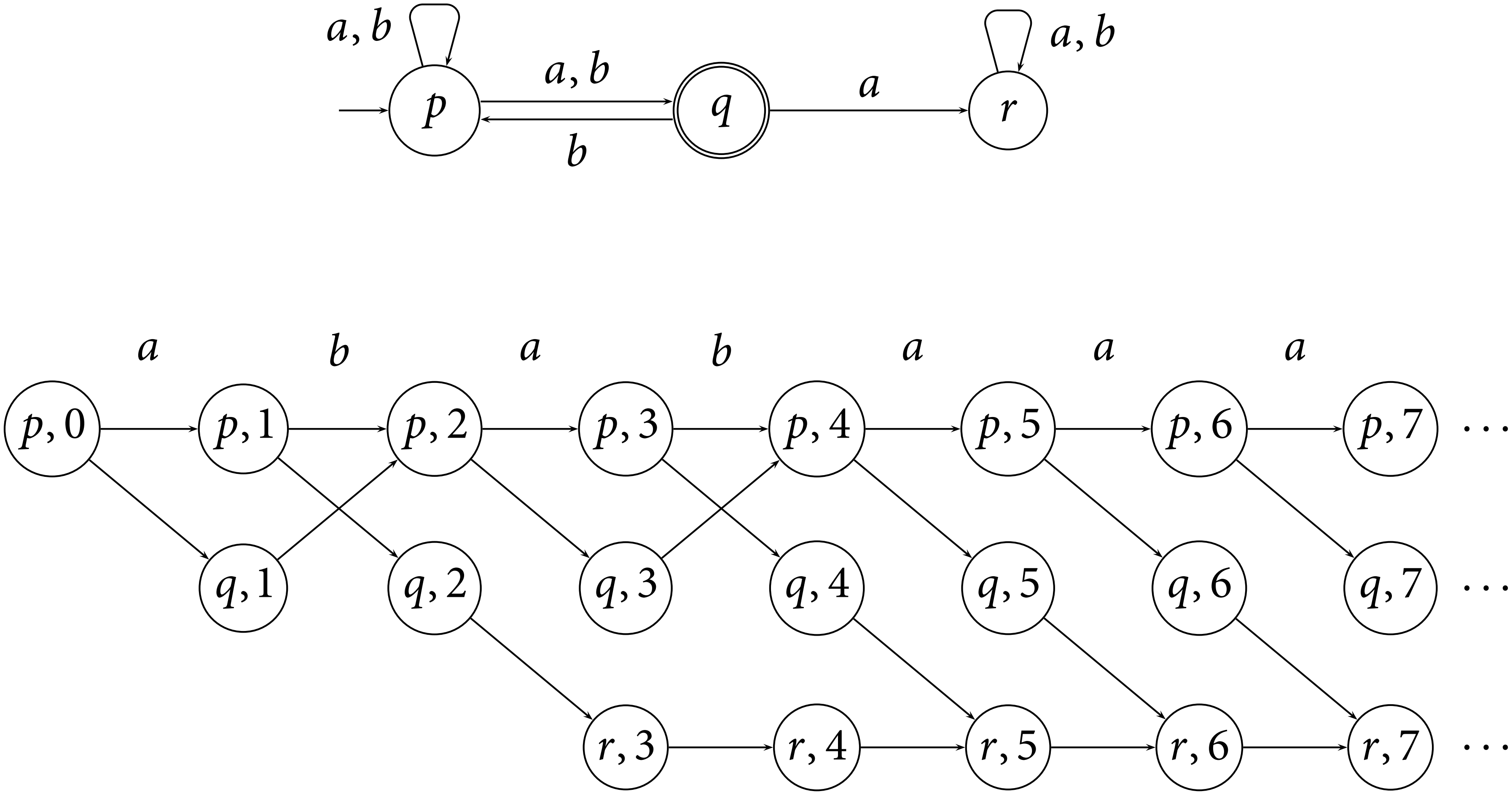
Lemma 1.

If there exists an odd ranking for G, then $\mathcal{A}$ does not accept α.





rank 1 --- rank 2 --- rank 3 --- rank 4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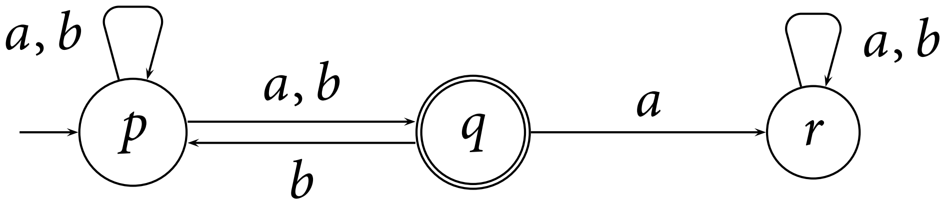
$$G = G_0 = G_1$$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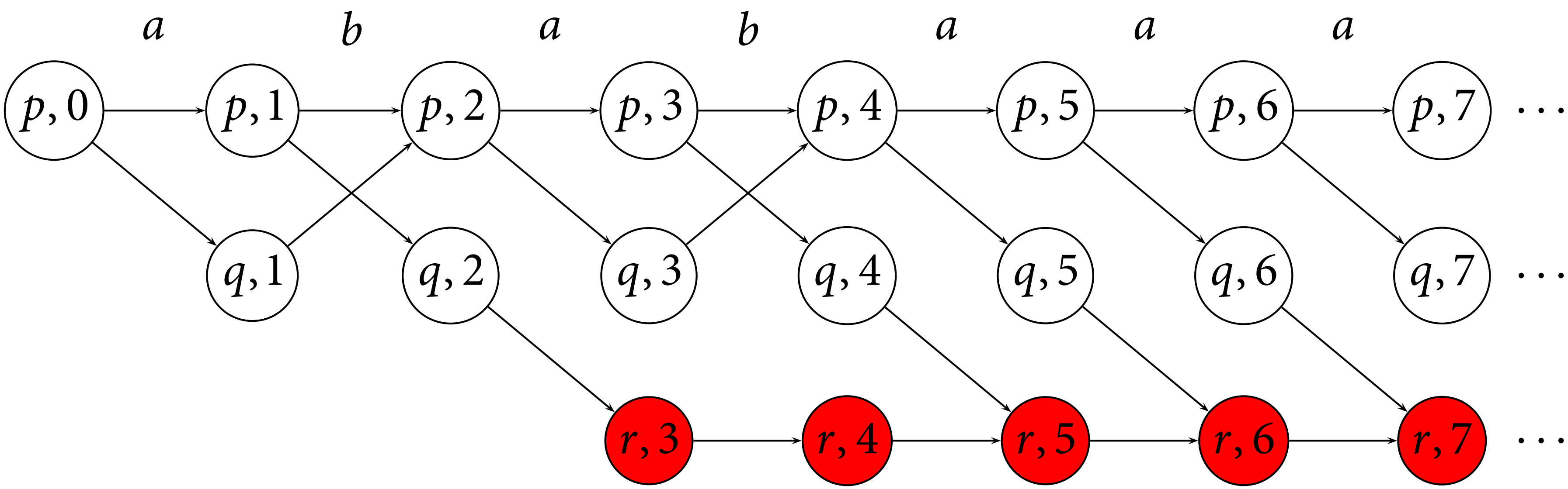
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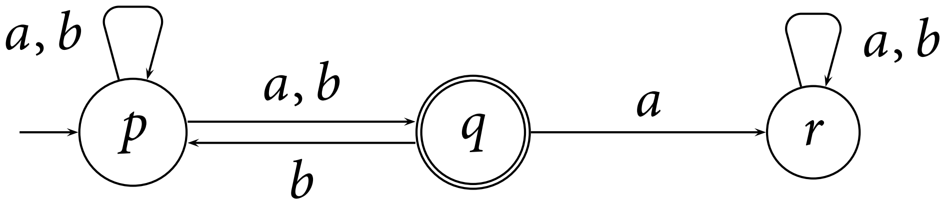
G_1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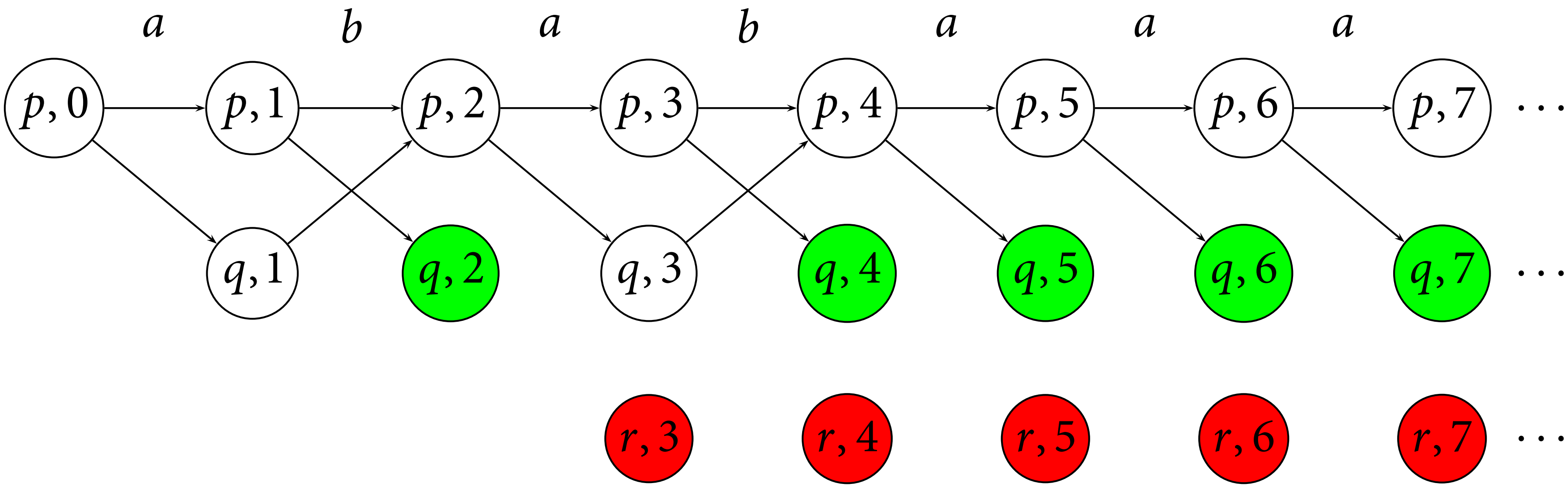
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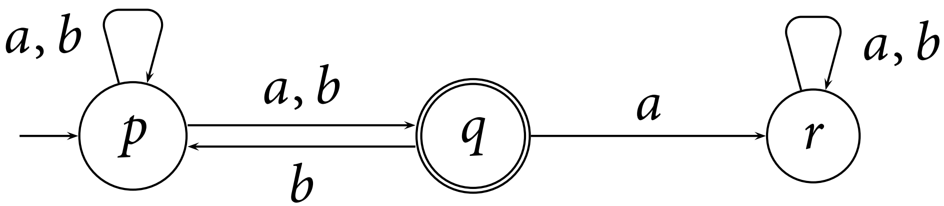
G_2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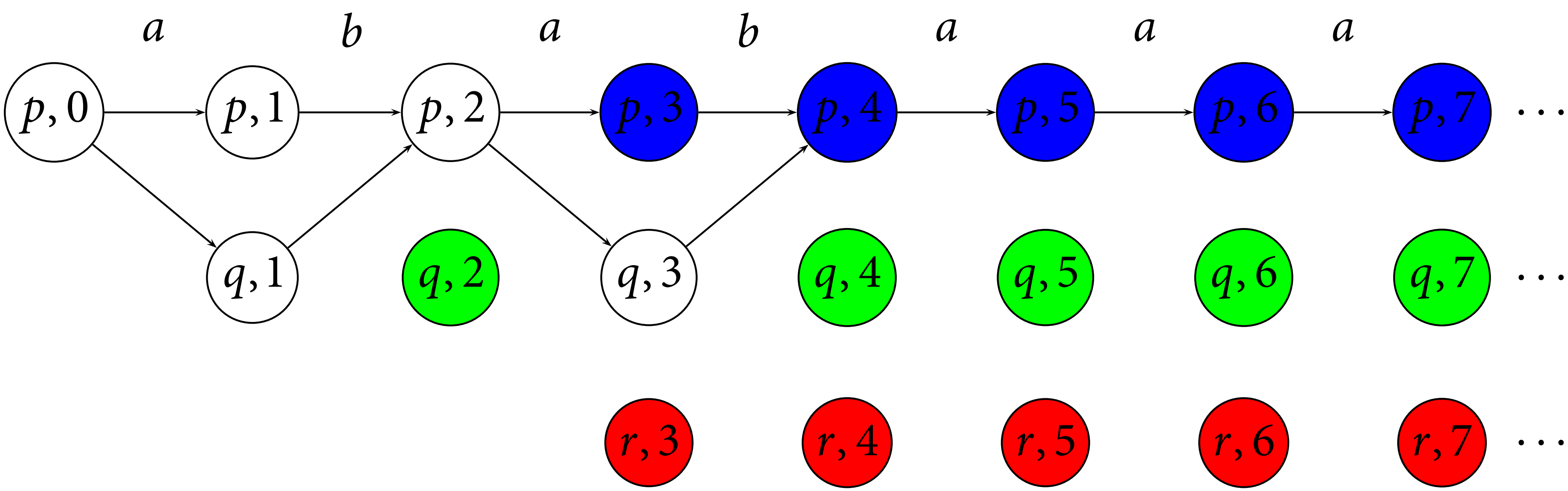
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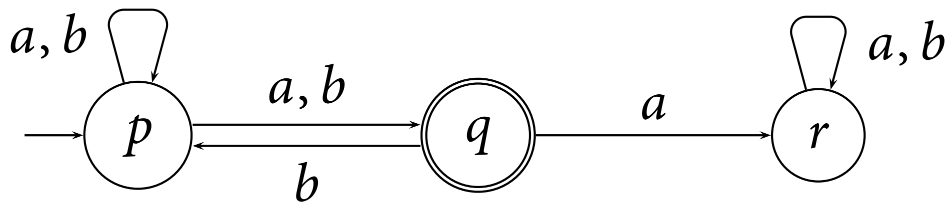
G_3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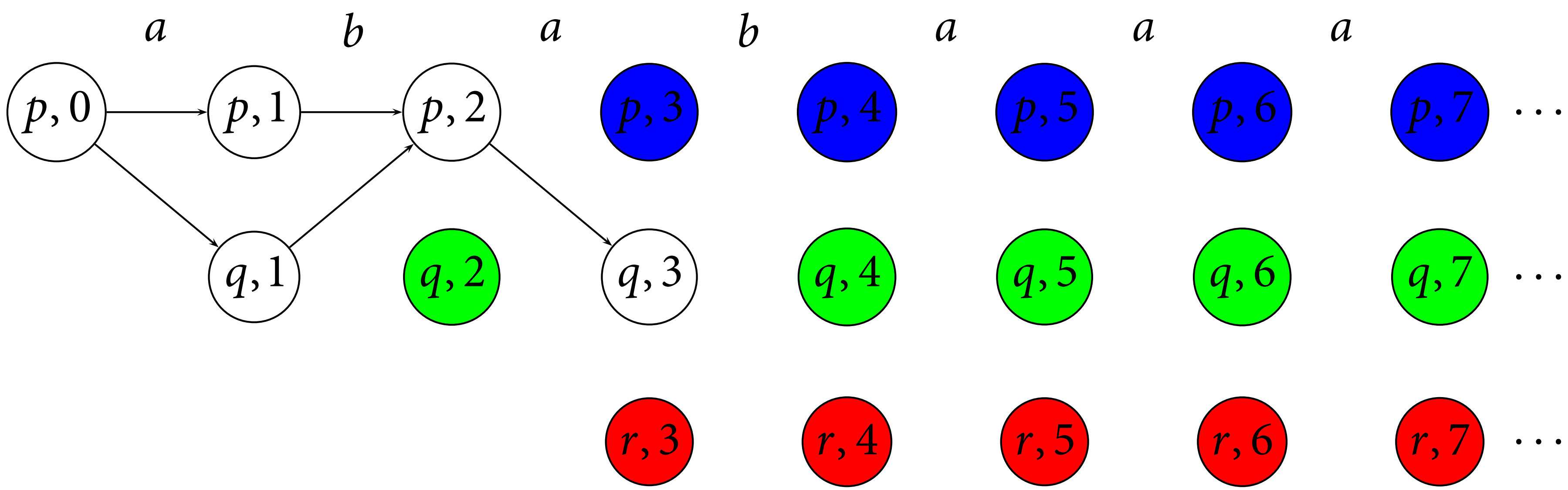
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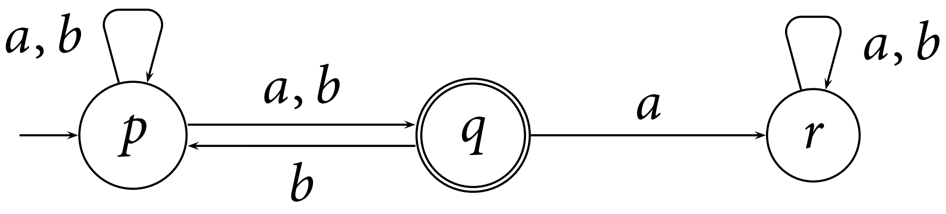
G_4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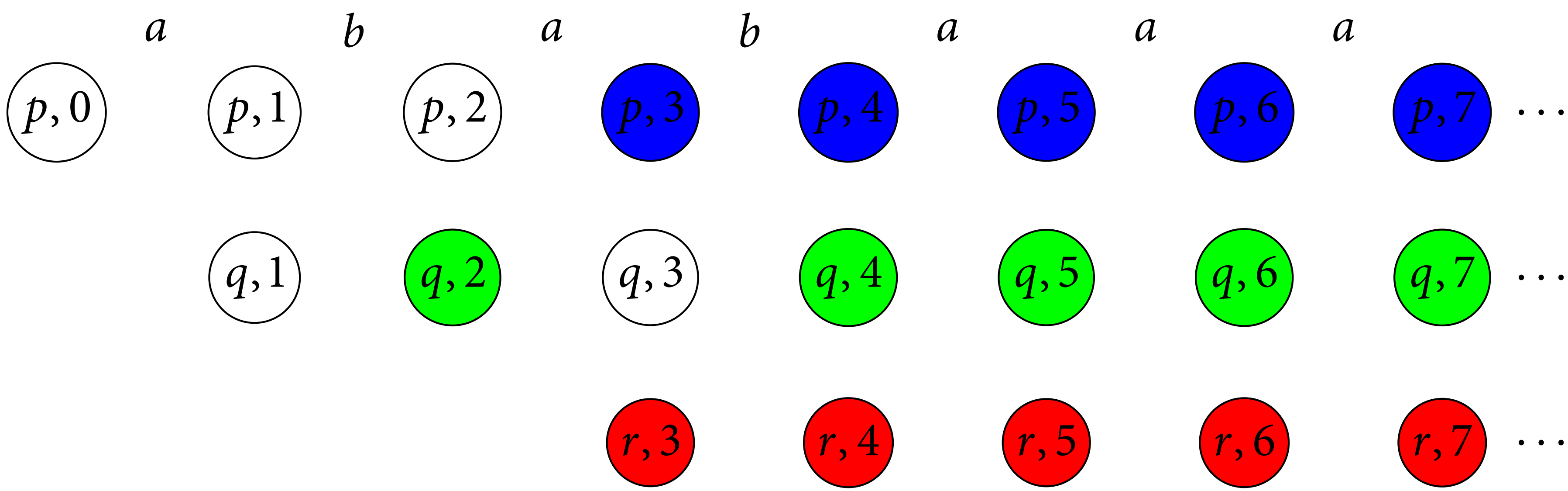
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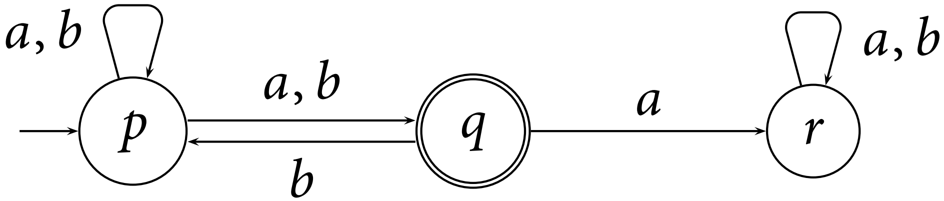
G_5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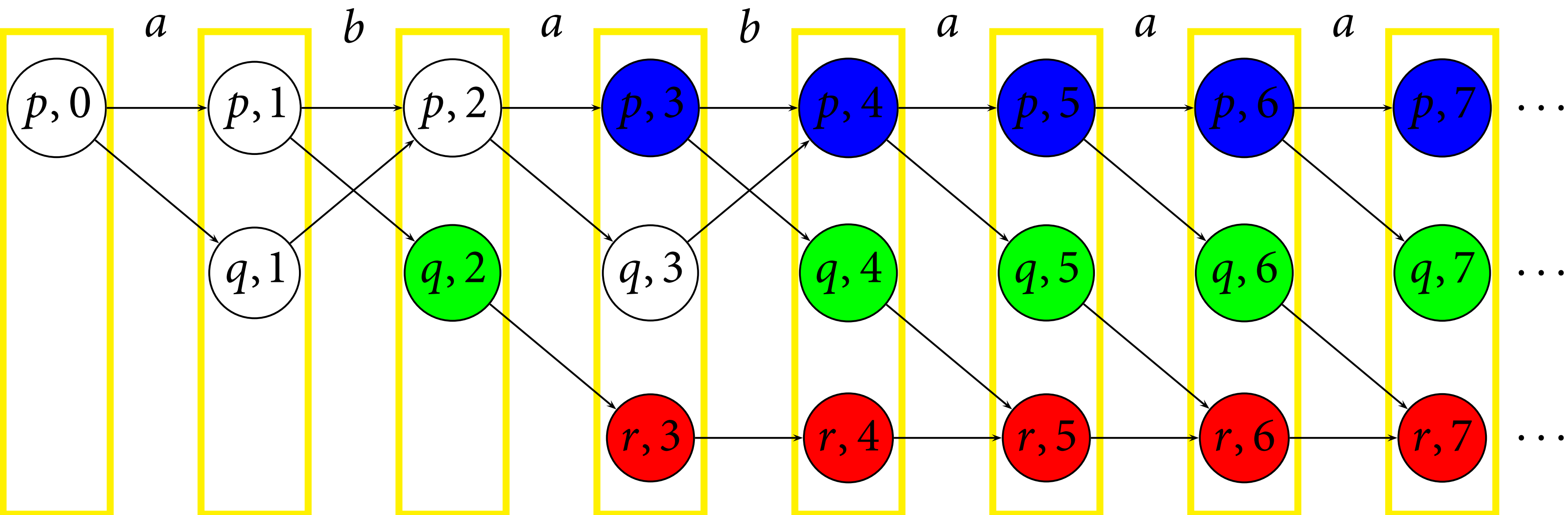
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