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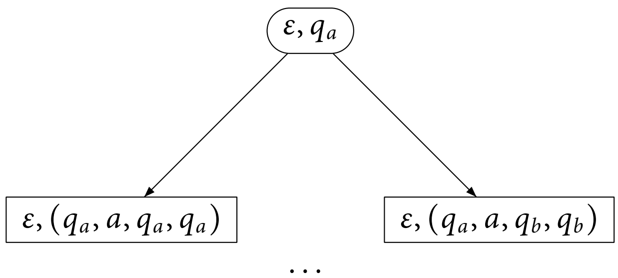
Automata, Games, and Verification: Lecture 12

Theorem 1 *A parity tree automaton $\mathcal{A} = (S, s_0, M, c)$ accepts an input tree t iff Player 0 wins the parity game $\mathcal{G}_{\mathcal{A},t} = (V_0, V_1, E, c')$ from position (ε, s_0).*

- $V_0 = \{(w,q) \mid w \in \{0,1\}^*, q \in S\}$;
- $V_1 = \{(w,\tau) \mid w \in \{0,1\}^*, \tau \in M\}$;
- $E = \{((w,q),(w,\tau)) \mid \tau = (q, t(w), q'_0, q'_1), \tau \in M\}$
  $\cup \{((w,\tau),(w',q')) \mid \tau = (q,\sigma,q'_0,q'_1)$ *and*
  $((w' = w0$ *and* $q' = q'_0)$ *or* $(w' = w1$ *and* $q' = q'_1))\}$;
- $c'(w,q) = c(q)$ *if* $q \in S$;
- $c'(w,\tau) = 0$ *if* $\tau \in M$.

Example:



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Proof:

- Given an accepting run r construct a winning strategy f_0:

$$f_0(w,q) = (w, (r(w), t(w), r(w0), r(w1)))$$

- Given a memoryless winning strategy f_0 construct an accepting run $r(\varepsilon) = s_0$ $\forall w \in \{0,1\}^*$
  - $r(w0) = q$ where $f_0(w, r(w)) = (w, (\_, \_, q, \_))$
  - $r(w1) = q$ where $f_0(w, r(w)) = (w, (\_, \_, \_, q))$

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Lemma 1 *For each parity tree automaton $\mathcal{A}$ over Σ-trees there exists a parity tree automaton $\mathcal{A}'$ over $\{1\}$-trees, such that $\mathcal{L}(\mathcal{A}) = \varnothing$ iff $\mathcal{L}(\mathcal{A}') = \varnothing$.*

Proof:

- $S' = S$;
- $s'_0 = s_0$;
- $M' = \{(q, 1, q_0.q_1) \mid (q, \sigma, q_0, q_1) \in M, \sigma \in \Sigma\}$
- $c' = c$

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Theorem 2 *The language of a parity tree automaton $\mathcal{A} = (S, s_0, M, c)$ is non-empty iff Player 0 wins the parity game $\mathcal{G}_{\mathcal{A},t} = (V_0, V_1, E, c')$ from position s_0.*

- $V_0 = S$;
- $V_1 = M$;
- $E = \{(q, \tau) \mid \tau = (q, 1, q'_0, q'_1), \tau \in M\}$
  $\cup \{(\tau, q') \mid \tau = (q, 1, q'_0, q'_1)$ *and*
  $(q' = q'_0$ *or* $q' = q'_1)\}$;
- $c'(q) = c(q)$ *for* $q \in S$;
- $c(\tau) = 0$ *for* $\tau \in M$.

Theorem 3 *Büchi tree automata are strictly weaker than parity tree automata.*

Proof:

- Consider the tree language $T = \{t \in T_{\{a,b\}} \mid$ every branch of t has only finitely many $b\}$
- T is recognized by a parity tree automaton. For example by $\mathcal{A} = (S, s_0, M, c)$ with $S = \{q_a, q_b\}$; $s_0 = q_a$; $M = \{(q_a, a, q_a, q_a), (q_b, a, q_a, q_a), (q_a, b, q_b, q_b), (q_b, b, q_b, q_b)\}$; $c(q_a) = 0, c(q_b) = 1$.
- T is not recognized by any Büchi tree automaton. Assume, by way of contradiction, that there is a Büchi tree automaton $\mathcal{A} = (S, s_0, M, F)$ such that $\mathcal{L}(\mathcal{A}) = T$.
  - Let $n = |S|$.
  - Consider the input tree t_n, where b appears exactly at nodes $1^+0, 1^+01^+0, \dots, (1^+0)^n$.
  - $t_n \in T \Rightarrow$ there exists an accepting run r of $\mathcal{A}$ on t_n.
  - On the branch consisting of the finite prefixes of 1^ω there are infinitely many visits to $F \Rightarrow \exists m_0 \in \omega$ such that $r(1^{m_0}) \in F$.
  - Analogously, on the branch consisting of the finite prefixes of $1^{m_0}01^\omega$, there are infinitely many visits to $F \Rightarrow \exists m_1 \in \omega$ such that $r(1^{m_0}01^{m_1}) \in F$.
  - Repeating this argument, we obtain $n+1$ positions $1^{m_0}, 1^{m_0}01^{m_1}, \dots, 1^{m_0}01^{m_1}0\dots01^{m_n}$ where F is visited.

- There must exist two different nodes u, v on the path to $1^{m_0}01^{m_1}0\dots01^{m_n}$ such that u is a prefix of v and $r(u) = r(v) \in F$. The path from u to v contains a left turn and therefore contains a node labeled with b.
- We construct a new input tree t_n and a run tree r' by repeating the path from u to v infinitely often:
  * let $v = u \cdot \pi$.
  * $t'_n(x) = t_n(u \cdot y)$ if $x = u \cdot \pi^* \cdot y$ for some shortest $y \in \{0,1\}^*$
    $t'_n(x) = t_n(x)$ otherwise
  * $r'(x) = r(u \cdot y)$ if $x = u \cdot \pi^* \cdot y$ for some shortest $y \in \{0,1\}^*$
    $r'(x) = r(x)$ otherwise
  * r' is accepting: the branch consisting of the finite prefixes of $u \cdot \pi^\omega$ has infinitely many visits to F; all other branches have the same labeling as in r after some finite prefix. Since r is accepting, these branches thus must also visit F infinitely often.
  * Hence t'_n is accepted by $\mathcal{A}$, but $t'_n \notin T$, because the branch consisting of the finite prefixes of $u \cdot \pi^\omega$ has infinitely many bs. Contradiction.

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17 Complementation of Parity Tree Automata

Reference: W. Thomas: *Languages, Automata and Logic,* Handbook of formal languages, Volume 3.

Theorem 4 *For each parity tree automaton $\mathcal{A}$ over Σ there is a parity tree automaton $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}') = T_\Sigma - \mathcal{L}(\mathcal{A})$.*

Proof:

- $\mathcal{A}$ does *not* accept some tree t iff Player 1 has a winning memoryless strategy f in $\mathcal{G}_{\mathcal{A},t}$ from (ε, s_0)
- Strategy
  $$f : \{0,1\}^* \times M \to \{0,1\}^* \times S$$
  can be represented as
  $$f' : \{0,1\}^* \times M \to \{0,1\}$$
  (where $f(u,(q,\sigma,q'_0,q'_1)) = (u \cdot i, q'_i)$ iff $f'(u,\tau) = i$).
- f' is isomorphic to
  $$g : \{0,1\}^* \to (M \to \{0,1\})$$
  ($M \to \{0,1\}$ is the finite “local strategy”)
- Hence, $\mathcal{A}$ does not accept t iff

(1) there is a $(M \to \{0,1\})$-tree v such that

(2) for all $i_0, i_1, i_2, \ldots \in \{0,1\}^\omega$

(3) for all $\tau_0, \tau_1, \ldots \in M^\omega$

(4) <u>if</u>

- for all j,
  $\tau_j = (q, a, q'_0, q'_1)$
  $\Rightarrow a = t(i_0, i_1, \ldots, i_j)$ and
- $i_0 i_1 \ldots = v(\varepsilon)(\tau_0) v(i_0)(\tau_1) \ldots$

<u>then</u> the generated state sequence $q_0 q_1 \ldots$
with $q_0 = s_0$, $(q_j, a, q^0, q^1) = \tau_j$,
$q_{j+1} = q^{v(i_0,\ldots,i_{j-1})(\tau_j)}$ for all j
violates c.

- to be continued.

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