Bernd Finkbeiner Date: October 25, 2012

Automata, Games, and Verification: Lecture 2

2 Büchi Automata

Definition 1 *A* nondeterministic Büchi automaton $\mathcal{A}$ *over alphabet* Σ *is a tuple* (S, I, T, F)*:*

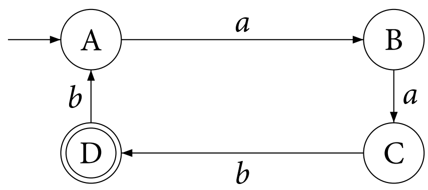
- S *: a finite set of* states
- $I \subseteq S$ *: a subset of* initial states
- $T \subseteq S \times \Sigma \times S$ *: a set of* transitions
- $F \subseteq S$ *: a subset of* accepting states

Now we define how a Büchi automaton uses an infinite word as input. Notice that we do not refer to acceptance in this definition.

Definition 2 *A* run *of a nondeterministic Büchi automaton* $\mathcal{A}$ *on an infinite input word* $\alpha = \sigma_0\sigma_1\sigma_2\ldots$ *is an infinite sequence of states* $s_0, s_1, s_2, \ldots$ *such that the following hold:*

- $s_0 \in I$
- *for all* $i \in \omega$, $(s_i, \sigma_i, s_{i+1}) \in T$

Example:



In the automaton shown the set of states are $S = \{A, B, C, D\}$, the initial set of states are $I = \{A\}$ (indicated with pointing arrow with no source), the transitions $T = \{(A, a, B), (B, a, C), (C, b, D), (D, b, A)\}$ are the remaining arrows in the diagram, and the set of accepting states is $F = \{D\}$ (double-lined state circle).

On input $aabbaabb\ldots$ the Büchi automaton shown has only the run:

$ABCDABCDABCD\ldots$ ✚

Determinism is a property of machines that can only react in a unique way to their input. The following definition makes this clear for Büchi automata.

Definition 3 *A Büchi automaton $\mathcal{A}$ is* deterministic *when T is a partial function (with respect to the next input letter and the current state):*

$\forall \sigma \in \Sigma, \forall s, s_0, s_1 \in S \,.\, (s, \sigma, s_0) \in T$ *and* $(s, \sigma, s_1) \in T \;\Rightarrow\; s_0 = s_1$

and I is singleton.

(By Büchi automaton we usually mean nondeterministic Büchi automaton.)

Definition 4 *The* infinity set *of an infinite word $\alpha \in \Upsilon^\omega$ over some alphabet Υ is the set $In(\alpha) = \{v \in \Upsilon \mid \forall i \exists j \,.\, j \geq i \text{ and } \alpha(j) = v\}$*

Definition 5 *A Büchi automaton $\mathcal{A}$* accepts *an infinite word α if:*

- *there is a run $r = s_0 s_1 s_2 \ldots$ of α on $\mathcal{A}$*
- *r is* accepting*: $In(r) \cap F \neq \varnothing$*

The language recognized by Büchi automaton $\mathcal{A}$ *is defined as follows:*

$\mathcal{L}(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \alpha\}$

Example: The automaton from the previous example has the language $\{aabbaabbaabb\ldots\}$. ✚

Comment: A deterministic Büchi automaton $\mathcal{A} = (S, I, T, F)$ defines a partial function[1] from Σ^ω to a set of runs $R \subseteq S^\omega$. ⬣

Definition 6 *An ω-language L is Büchi recognizable if there is a Büchi automaton $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = L$.*

Example: The singleton ω-language $L = \{\sigma\}$ with $\sigma = abaabaaabaaaab\ldots$ is not Büchi recognizable. (Note that all finite languages of finite words are NFA-recognizable. Analog result does not hold for Büchi-automata)

Proof:

- Suppose there is a Büchi automaton $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) = L$.
- Let $r = s_0 s_1 \ldots$ be an accepting run on σ.
- Since F is finite, there exists $k, k' \in \omega$ with $k < k'$ and $s_k = s_{k'} \in F$.
- $r' = r_0 \ldots r_{k'-1}(r_k \ldots r_{k'-1})^\omega$ is an accepting run on $\sigma' = \sigma(0) \ldots \sigma(k'-1)(\sigma(k) \ldots \sigma(k'-1))^\omega$.
- Hence, $\sigma' \in \mathcal{L}(\mathcal{A})$. Contradiction.

■

✚

Definition 7 *A Büchi automaton is* complete *if its transition relation contains a function:*

$\forall s \in S, \sigma \in \Sigma \,.\, \exists s' \in S \,.\, (s, \sigma, s') \in T$

[1] A *partial function* is a function that is not defined on all of the elements of its domain.

Theorem 1 *For every Büchi automaton $\mathcal{A}$, there is a complete Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

We define $\mathcal{A}'$ in terms of the components S, I, T, F of $\mathcal{A}$:

$$S' = S \cup \{f\} \qquad f \text{ new}$$

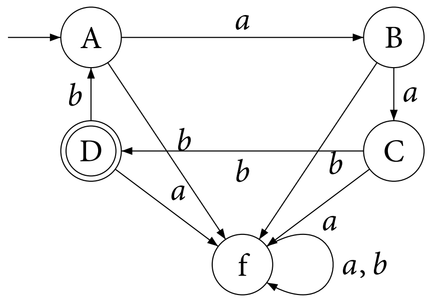
$$I' = I$$

$$T' = T \cup \{(s, \sigma, f) \mid \nexists s' . (s, \sigma, s') \in T\} \cup \{(f, \sigma, f) \mid \sigma \in \Sigma\}$$

$$F' = F$$

The runs of $\mathcal{A}'$ are a superset of those of $\mathcal{A}$ since we have added states and transitions. Furthermore, on any infinite input word α the accepting runs of $\mathcal{A}$ and $\mathcal{A}'$ correspond, because any run that reaches f stays in f, and since $f \notin F'$, such a run is not accepting. ■

Example: Completing the automaton from the previous examples we obtain the following automaton:



✚

Unless we specify otherwise, we will only consider complete automata when we prove results.

Comment: A complete deterministic Büchi automaton $\mathcal{A} = (S, I, T, F)$ may be viewed as a total function from Σ^ω to S^ω. A complete (possibly nondeterministic) Büchi automaton produces at least one run for every Σ^ω input word. ⬢

3 ω-regular Languages

Definition 8 *The* ω-regular expressions *are defined as follows.*

- *If R is an regular expression where $\varepsilon \notin \mathcal{L}(R)$, then R^ω is an ω-regular expression.* $\mathcal{L}(R^\omega) = \mathcal{L}(R)^\omega$ *where $L^\omega = \{u_0 u_1 \dots \mid u_i \in L, |u_i| > 0$ for all $i \in \omega\}$ for $L \subseteq \Sigma^*$.*

- *If R is a regular expression and U is an ω-regular expression, then $R \cdot U$ is an ω-regular expression.*
  $\mathcal{L}(R \cdot U) = \mathcal{L}(R) \cdot \mathcal{L}(U)$
  where $L_1 \cdot L_2 = \{r \cdot u \mid r \in L_1, u \in L_2\}$ *for* $L_1 \subseteq \Sigma^*, L_2 \subseteq \Sigma^\omega$.

- *If U_1 and U_2 are ω-regular expressions, then $U_1 + U_2$ is an ω-regular expression.*
  $\mathcal{L}(U_1 + U_2) = \mathcal{L}(U_1) \cup \mathcal{L}(U_2)$.

Definition 9 *An ω-*regular *language is a finite union of ω-languages of the form $U \cdot V^\omega$ where $U, V \subseteq \Sigma^*$ are regular languages.*

Theorem 2 *If L_1 and L_2 are Büchi recognizable, then so is $L_1 \cup L_2$.*

Proof:

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Büchi automata that recognize L_1 and L_2, respectively. We construct an automaton $\mathcal{A}'$ for $L_1 \cup L_2$:

- $S' = S_1 \cup S_2$ (w.l.o.g. we assume $S_1 \cap S_2 = \emptyset$);
- $I' = I_1 \cup I_2$;
- $T' = T_1 \cup T_2$;
- $F' = F_1 \cup F_2$.

$\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$: For $\alpha \in \mathcal{L}(A')$, we have an accepting run $r = s_0 s_1 s_2 \dots$ of α in $\mathcal{A}'$. If $s_0 \in S_1$, then r is an accepting run on $\mathcal{A}_1$, otherwise $s_0 \in S_2$ and r is an accepting run on $\mathcal{A}_2$.

$\mathcal{L}(\mathcal{A}') \supseteq \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$: For $i \in \{1, 2\}$ and $\alpha \in \mathcal{L}(\mathcal{A}_i)$, there is an accepting run $r = s_0 s_1 s_2 \dots$ on $\mathcal{A}_i$. The run r is accepting for α in $\mathcal{A}'$. ■

Theorem 3 *If L_1 and L_2 are Büchi recognizable, then so is $L_1 \cap L_2$.*

Proof:

We construct an automaton $\mathcal{A}'$ from $\mathcal{A}_1$ and $\mathcal{A}_2$:

- $S' = S_1 \times S_2 \times \{1, 2\}$
- $I' = I_1 \times I_2 \times \{1\}$
- $T' = \{((s_1, s_2, 1), \sigma, (s_1', s_2', 1)) \mid (s_1, \sigma, s_1') \in T_1, (s_2, \sigma, s_2') \in T_2, s_1 \notin F_1\}$
  $\cup \{((s_1, s_2, 1), \sigma, (s_1', s_2', 2)) \mid (s_1, \sigma, s_1') \in T_1, (s_2, \sigma, s_2') \in T_2, s_1 \in F_1\}$
  $\cup \{((s_1, s_2, 2), \sigma, (s_1', s_2', 2)) \mid (s_1, \sigma, s_1') \in T_1, (s_2, \sigma, s_2') \in T_2, s_2 \notin F_2\}$
  $\cup \{((s_1, s_2, 2), \sigma, (s_1', s_2', 1)) \mid (s_1, \sigma, s_1') \in T_1, (s_2, \sigma, s_2') \in T_2, s_2 \in F_2\}$
- $F' = \{(s_1, s_2, 2) \mid s_1 \in S_1, s_2 \in F_2\}$

$\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$:

- $r' = (s_1^0, s_2^0, t^0)(s_1^1, s_2^1, t^1)\dots$ is a run of $\mathcal{A}'$ on input word σ iff $r_1 = s_1^0 s_1^1 \dots$ is a run of $\mathcal{A}_1$ on σ and $r_2 = s_2^0 s_2^1 \dots$ is a run of $\mathcal{A}_2$ on σ.

- r' is accepting iff r_1 is accepting and r_2 is accepting.

■

Theorem 4 *If L_1 is a regular language and L_2 is Büchi recognizable, then $L_1 \cdot L_2$ is Büchi-recognizable.*

Proof:

Let $\mathcal{A}_1$ be a finite-word automaton that recognizes L_1 and $\mathcal{A}_2$ be a Büchi automaton that recognizes L_2. We construct:

- $S' = S_1 \cup S_2$ (w.l.o.g. we assume $S_1 \cap S_2 = \emptyset$);
- $I' = \begin{cases} I_1 & \text{if } I_1 \cap F_1 = \emptyset \\ I_1 \cup I_2 & \text{otherwise;} \end{cases}$
- $T' = T_1 \cup T_2 \cup \{(s, \sigma, s') \mid (s, \sigma, f) \in T_1, f \in F_1, s' \in I_2\}$;
- $F' = F_2$.

■

Theorem 5 *If L is a regular language then L^ω is Büchi recognizable.*

Proof:

Let $\mathcal{A}$ be a finite word automaton; let w.l.o.g. $\varepsilon \notin \mathcal{L}(\mathcal{A})$.

- **Step 1:** Ensure that all initial states have no incoming transitions. We modify $\mathcal{A}$ as follows:
  - $S' = S \cup \{s_{\text{new}}\}$;
  - $I' = \{s_{\text{new}}\}$;
  - $T' = T \cup \{(s_{\text{new}}, \sigma, s') \mid (s, \sigma, s') \in T \text{ for some } s \in I\}$;
  - $F' = F$.

  This modification does not affect the language of $\mathcal{A}$.
- **Step 2:** Add loop:
  - $S'' = S'$; $I'' = I'$;
  - $T'' = T' \cup \{(s, \sigma, s_{\text{new}} \mid (s, \sigma, s') \in T' \text{ and } s' \in F'\}$;
  - $F'' = I'$.

$\mathcal{L}(\mathcal{A}'') \subseteq \mathcal{L}(\mathcal{A}')^\omega$:

- Assume $\alpha \in \mathcal{L}(\mathcal{A}'')$ and $s_0 s_1 s_2 \ldots$ is an accepting run for α in $\mathcal{A}''$.
- Hence, $s_i = s_{\text{new}} \in F'' = I'$ for infinitely many indices i: $i_0, i_1, i_2, \ldots$.
- This provides a series of runs in $\mathcal{A}'$:
  - run $s_0 s_1 \ldots s_{i_1 - 1} s$ on $w_1 = \alpha(0)\alpha(1)\ldots\alpha(i_1 - 1)$ for some $s \in F'$;
  - run $s_{i_1} s_{i_1+1} \ldots s_{i_2 - 1} s$ on $w_2 = \alpha(i_1)\alpha(i_1 + 1)\ldots\alpha(i_2 - 1)$ for some $s \in F'$;

- ...

- This yields $w_k \in \mathcal{L}(\mathcal{A}')$ for every $k \geq 1$.
- Hence, $\alpha \in \mathcal{L}(\mathcal{A}')^\omega$.

$\mathcal{L}(\mathcal{A}'') \supseteq \mathcal{L}(\mathcal{A}')^\omega$:

- Let $\alpha = w_1 w_2 w_3 \in \Sigma^\omega$ such that $w_k \in \mathcal{L}(\mathcal{A}')$ for all $k \geq 1$.
- For each k, we choose an accepting run $s_0^k s_1^k s_2^k \dots s_{n_k}^k$ of $\mathcal{A}'$ on w_k.
- Hence, $s_0^k \in I'$ and $s_{n_k}^k \in F'$ for all $k \geq 1$.
- Thus,
$$s_0^1 \dots s_{n_1-1}^1 s_0^2 \dots s_{n_2-1}^2 s_0^3 \dots s_{n_3-1}^3 \dots$$
is an accepting run on α in $\mathcal{A}''$.
- Hence, $\alpha \in \mathcal{L}(\mathcal{A}'')$.

■

Theorem 6 (Büchi's Characterization Theorem (1962)) *An ω-language is Büchi recognizable iff it is ω-regular.*