Bernd Finkbeiner Date: October 30, 2012

Automata, Games, and Verification: Lecture 3

Theorem 1 (Büchi's Characterization Theorem (1962)) *An ω-language is Büchi recognizable iff it is ω-regular.*

Proof:

"$\Leftarrow$" follows from previous constructions.

"$\Rightarrow$": Given a Büchi automaton $\mathcal{A}$, we consider for each pair $s, s' \in S$ the regular language

$$W_{s,s'} = \{u \in \Sigma^* \mid \text{finite-word automaton } (S, \{s\}, T, \{s'\}) \text{ accepts } u\} .$$

Claim: $\mathcal{L}(\mathcal{A}) = \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot W_{s',s'}{}^{\omega}$.

$\mathcal{L}(\mathcal{A}) \subseteq \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot W_{s',s'}{}^{\omega}$:

- Let $\alpha \in \mathcal{L}(\mathcal{A})$.
- Then there is an accepting run r for α on $\mathcal{A}$, which begins at some $s \in I$ and visits some $s' \in F$ infinitely often:

  $$r : s \stackrel{\alpha_0}{\to} s' \stackrel{\alpha_1}{\to} s' \stackrel{\alpha_2}{\to} s' \stackrel{\alpha_3}{\to} s' \stackrel{\alpha_4}{\to} s' \to \ \dots,$$

  where $\alpha = \alpha_0 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \cdot \dots$.
  (Notation:
  $s_0 \stackrel{\sigma_0 \sigma_1, \dots \sigma_k}{\longrightarrow} s_{k+1}$: there exist $s_1, \dots s_k$ s.t. $(s_i, \sigma_i, s_{i+1}) \in$ for all $0 \leq i \leq k$.)
- Hence, $\alpha_0 \in W_{s,s'}$ and $\alpha_k \in W_{s',s'}$ for $k > 0$ and thus $\alpha \in W_{s,s'} \cdot W_{s',s'}{}^{\omega}$ for some $s \in I, s' \in F$.

$\mathcal{L}(\mathcal{A}) \supseteq \bigcup_{s \in I, s' \in F} W_{s,s'} \cdot W_{s',s'}{}^{\omega}$:

- Let $\alpha = \alpha_0 \cdot \alpha_1 \cdot \alpha_2 \cdot \dots$ with $\alpha_0 \in W_{s,s'}$ and $\alpha_k \in W_{s',s'}$ for some $s \in I, s' \in F$.
- Then the run

  $$r : s \stackrel{\alpha_0}{\to} s' \stackrel{\alpha_1}{\to} s' \stackrel{\alpha_2}{\to} s' \stackrel{\alpha_3}{\to} s' \stackrel{\alpha_4}{\to} s' \to$$

  exists and is accepting since $s' \in F$.
- It follows that $\alpha \in \mathcal{L}(\mathcal{A})$.

■

4 Deterministic Büchi Automata

Theorem 2 *The language $(a+b)^* b^\omega$ is not recognizable by a deterministic Büchi automaton.*

Proof:

- Assume that L is recognized by the deterministic Büchi automaton $\mathcal{A}$.
- Since $b^\omega \in L$, there is a run
  $r_0 = s_{0,0}s_{0,1}s_{0,2}, \ldots$
  with $s_{0,n_0} \in F$ for some $n_0 \in \omega$.
- Similarly, $b^{n_0}ab^\omega \in L$ and there must be a run
  $r_1 = s_{0,0}s_{0,1}s_{0,2} \ldots s_{0,n_0}s_1 s_{1,0}s_{1,1}s_{1,2} \ldots$
  with $s_{1,n_1} \in F$
- Repeating this argument, there is a word $b^{n_0}ab^{n_1}ab^{n_2}a \ldots$ accepted by $\mathcal{A}$.
- This contradicts $L = \mathcal{L}(\mathcal{A})$.

■

Definition 1 (Substrings) *Let $\alpha \in \Sigma^\omega$. For $n, m \in \omega, n \leq m$ we define*

$$\alpha(n,m) = \alpha(n)\alpha(n+1)\ldots\alpha(m) .$$

Definition 2 (Limit) *For $W \subseteq \Sigma^*$:*

$$\overrightarrow{W} = \{\alpha \in \Sigma^\omega \mid \text{there exist infinitely many } n \in \omega \text{ s.t. } \alpha(0,n) \in W\} .$$

Theorem 3 *An ω-language $L \subseteq \Sigma^\omega$ is recognizable by a deterministic Büchi automaton iff there is a regular language $W \subseteq \Sigma^*$ s.t. $L = \overrightarrow{W}$.*

Proof:

Let L be the language of a deterministic Büchi automaton $\mathcal{A}$; let W be the regular language of $\mathcal{A}$ as a deterministic finite-word automaton. We show that $L = \overrightarrow{W}$.

$\alpha \in L$

iff for the unique run r of $\mathcal{A}$ on α, $In(r) \cap F \neq \emptyset$

iff $\alpha(0,n) \in W$ for infinitely many $n \in \omega$

iff $\alpha \in \overrightarrow{W}$.

■

Theorem 4 *For any deterministic Büchi automaton $\mathcal{A}$, there exists a Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.*

Proof:

We construct $\mathcal{A}'$ as follows:

- $S' = (S \times \{0\}) \cup ((S \setminus F) \times \{1\})$.
- $I' = I \times \{0\}$.
- $T' = \{((s,0),\sigma,(s',0)) \mid (s,\sigma,s') \in T\}$
$\cup \{((s,0),\sigma,(s',1)) \mid (s,\sigma,s') \in T, s' \in S - F\}$
$\cup \{((s,1),\sigma,(s,1)) \mid (s,\sigma,s') \in T, s' \in S - F\}$.
- $F' = (S - F) \times \{1\}$.

$\mathcal{L}(\mathcal{A}') \subseteq \Sigma^\omega - \mathcal{L}(\mathcal{A})$:

- For $\alpha \in \mathcal{L}(\mathcal{A}')$ we have an accepting run

$$r' : (s_0,0)(s_1,0)\ldots(s_j,0)(s_0',1)(s_1',1)\ldots$$

on $\mathcal{A}'$.
- Hence,

$$r : s_0 s_1 s_2 \ldots s_j s_0' s_1' \ldots$$

is the unique run on α in $\mathcal{A}$.
- Since $s_0', s_1', \ldots \in S \setminus F$, $In(r) \subseteq S \setminus F$. Hence, r is not accepting and $\alpha \in \Sigma^\omega - \mathcal{L}(\mathcal{A})$

$\mathcal{L}(\mathcal{A}') \supseteq \Sigma^\omega - \mathcal{L}(\mathcal{A})$:

- We assume $\alpha \notin \mathcal{L}(\mathcal{A})$. Since $\mathcal{A}$ is deterministic and complete there exists a run

$$r : s_0 s_1 s_2 \ldots$$

for α on $\mathcal{A}$, but $In(r) \cap F = \emptyset$.
- Thus there exists a $k \in \omega$ such that $s_j \notin F$ for $j > k$.
- This gives us the run

$$r' : (s_0,0)(s_1,0)\ldots(s_k,0)(s_{k+1},1)(s_{k+2},1)\ldots$$

for α on $\mathcal{A}'$ with the property $In(r') \subseteq ((S - F) \times \{1\}) = F'$.
- Hence, r' is accepting and $\alpha \in \mathcal{L}(\mathcal{A}')$.

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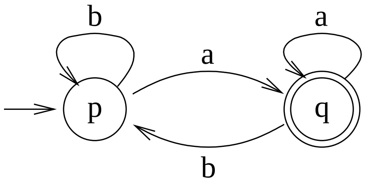
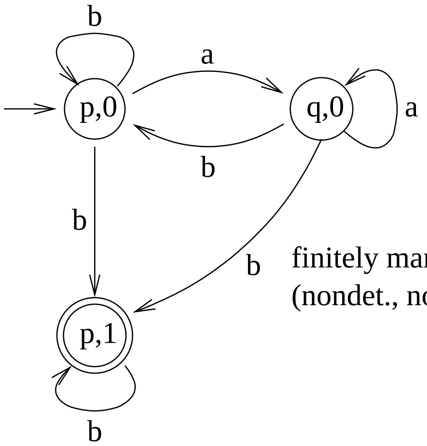
Example:



infinitely many 'a's



finitely many 'a's
(nondet., not complete)

✚

5 Complementation of Nondeterministic Büchi Automata

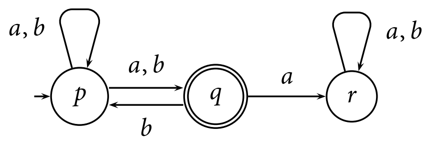
Reference: The following construction for the complementation of nondeterministic Büchi automata is taken from: Orna Kupferman and Moshe Y. Vardi, Weak alternating automata are not that weak. *ACM Trans. Comput. Logic* 2, 3 (Jul. 2001), 408-429.

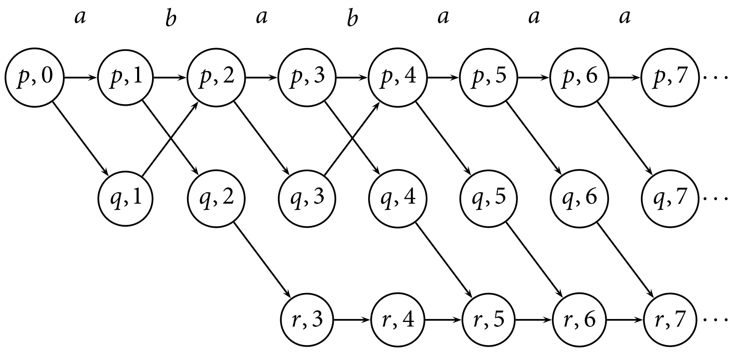
Definition 3 *Let* $\mathcal{A} = (S, I, T, F)$ *be a nondeterministic Büchi automaton. The* run DAG *of* $\mathcal{A}$ *on a word* $\alpha \in \Sigma^\omega$ *is the directed acyclic graph* $G = (V, E)$ *where*

- $V = \bigcup_{l \geq 0}(S_l \times \{l\})$ *where* $S_0 = I$ *and* $S_{l+1} = \bigcup_{s \in S_l, (s, \alpha(l), s') \in T}\{s'\}$
- $E = \{(\langle s, l\rangle, \langle s', l+1\rangle) \mid l \geq 0, (s, \alpha(l), s') \in T\}$

A path in a run DAG is accepting iff it visits $F \times \mathbb{N}$ infinitely often. The automaton accepts α if some path is accepting.

Example:





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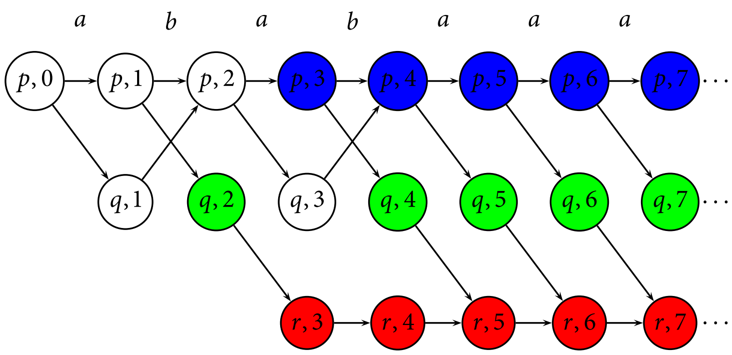
Definition 4 *A* ranking *for G is a function $f : V \to \{0, \ldots, 2 \cdot |S|\}$ such that*

- *for all $\langle s, l\rangle \in V$, if $f(\langle s, l\rangle)$ is odd then $s \notin F$;*
- *for all $(\langle s, l\rangle, \langle s', l'\rangle) \in E$, $f(\langle s', l'\rangle) \le f(\langle s, l\rangle)$.*

A ranking is *odd* iff for all paths $\langle s_0, l_0\rangle, \langle s_1, l_1\rangle, \langle s_2, l_2\rangle, \ldots$ in G, there is a $i \ge 0$ such that $f(\langle s_i, l_i\rangle)$ is odd and, for all $j \ge 0$, $f(\langle s_{i+j}, l_{i+j}\rangle) = f(\langle s_i, l_i\rangle)$.

Example:



rank 1 — rank 2 — rank 3 — rank 4

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Lemma 1 *If there exists an odd ranking for G, then $\mathcal{A}$ does not accept α.*

Proof:

- In an odd ranking, every path eventually gets trapped in a some odd rank.
- If $f(\langle s, l\rangle)$ is odd, then $s \notin F$.
- Hence, every path visits F only finitely often.

$\blacksquare$