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Automata, Games, and Verification: Lecture 4

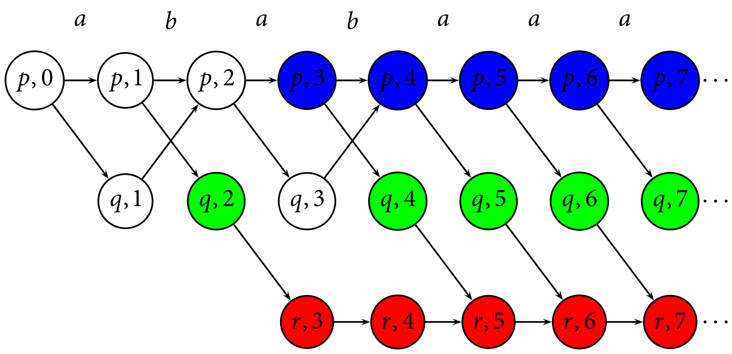
Let G' be a subgraph of G. We call a vertex $\langle s, l\rangle$

- *safe* in G' if for all vertices $\langle s', l'\rangle$ reachable from $\langle s, l\rangle$, $s' \notin F$, and
- *endangered* in G' if only finitely many vertices are reachable.

We define an infinite sequence $G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots$ of DAGs inductively as follows:

- $G_0 = G$
- $G_{2i+1} = G_{2i} \setminus \{\langle s, l\rangle \mid \langle s, l\rangle \text{ is endangered in } G_{2i}\}$
- $G_{2i+2} = G_{2i+1} \setminus \{\langle s, l\rangle \mid \langle s, l\rangle \text{ is safe in } G_{2i+1}\}$.

Example:



- no endangered vertices in G;
- safe in G_1;
- endangered in G_2;
- safe in G_3;
- all remaining vertices are endangered in G_4.

✚

Lemma 1 *If $\mathcal{A}$ does not accept α, then the following holds: For every $i \geq 0$ there exists an l_i such that for all $j \geq l_i$ at most $|S| - i$ vertices of the form $\langle \_, j\rangle$ are in G_{2i}.*

Proof:

Proof by induction on i:

- $i = 0$: In G, for every l, there are at most $|S|$ vertices of the form $\langle \_, l\rangle$.
- $i \to i + 1$:

- Case G_{2i} is finite: then $G_{2(i+1)}$ is empty.
- Case G_{2i} is infinite:
  * There must exist a safe vertex $\langle s, l \rangle$ in G_{2i+1}. (Otherwise, we can construct a path in G with infinitely many visits to F).
  * We choose $l_{i+1} = l$.
  * We prove that for all $j \geq l$, there are at most $|S| - (i+1)$ vertices of the form $\langle \_, j \rangle$ in G_{2i+2}.
    · Since $\langle s, l \rangle \in G_{2i+1}$, it is not endangered in G_{2i}.
    · Hence, there are infinitely many vertices reachable from $\langle s, l \rangle$ in G_{2i}.
    · By König's Lemma, there exists an infinite path $p = \langle s, l \rangle, \langle s_1, l+1 \rangle, \langle s, l+2 \rangle, \ldots$ in G_{2i}.
    · No vertex on p is endangered (there is an infinite path). Therefore, p is in G_{2i+1}.
    · All vertices on p are safe ($\langle s, l \rangle$ is safe) in G_{2i+1}. Therefore, none of the vertices on p are in G_{2i+2}.
    · Hence, for all $j \geq l$, the number of vertices of the form $\langle \_, l \rangle$ in G_{2i+2} is strictly smaller than their number in G_{2i}.

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Lemma 2 *If $\mathcal{A}$ does not accept α, then there exists an odd ranking for G.*

Proof:

- We define $f(\langle s, l \rangle) = 2i$ if $\langle s, l \rangle$ is endangered in G_{2i} and
- $f(\langle s, l \rangle) = 2i + 1$ if $\langle s, l \rangle$ is safe in G_{2i+1}.
- f is a ranking:
  - by Lemma 1, G_j is empty for $j > 2 \cdot |S|$. Hence, $f : V \to \{0, \ldots, 2 \cdot |S|\}$.
  - if $\langle s', l' \rangle$ is a successor of $\langle s, l \rangle$, then $f(\langle s', l' \rangle) \leq f(\langle s, l \rangle)$
    * Let $j := f(\langle s, l \rangle)$.
    * Case j is even: vertex $\langle s, l \rangle$ is endangered in G_j; hence either $\langle s', l' \rangle$ is not in G_j, and therefore $f(\langle s, l \rangle) < j$; or $\langle s', l' \rangle$ is in G_j and endangered; hence, $f(\langle s, l \rangle) = j$.
    * Case j is odd: vertex $\langle s, l \rangle$ is safe in G_j; hence either $\langle s', l' \rangle$ is not in G_j, and therefore $f(\langle s, l \rangle) < j$; or $\langle s', l' \rangle$ is in G_j and safe; hence, $f(\langle s, l \rangle) = j$.
  - f is an odd ranking:
    * For every path $\langle s_0, l_0 \rangle, \langle s_1, l_1 \rangle, \langle s_2, l_2 \rangle, \ldots$ in G there exists an $i \geq 0$ such that for all $j \geq 0$, $f(\langle s_{i+j}, l_{i+j} \rangle) = f(\langle s_i, l_i \rangle)$.
    * Suppose that $k := f(\langle s_i, l_i \rangle)$ is even. Thus, $\langle s_i, l_i \rangle$ is endangered in G_k.
    * Since $f(\langle s_{i+j}, l_{i+j} \rangle) = k$ for all $j \geq 0$, all $\langle s_{i+j}, l_{i+j} \rangle$ are in G_k.
    * This contradicts that $\langle s_i, l_i \rangle$ is endangered in G_k.

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Theorem 1 *For each Büchi automaton $\mathcal{A}$ there exists a Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$.*

Helpful definitions:

- A *level ranking* is a function $g : S \to \{0, \ldots, 2 \cdot |S|\} \cup \{\bot\}$ such that if $g(s)$ is odd, then $s \notin F$.
- Let $\mathcal{R}$ be the set of all level rankings.
- A level ranking g' *covers* a level ranking g if, for all $s, s' \in S$, if $g(s) \neq \bot$ and $(s, \sigma, s') \in T$, then $\bot \neq g'(s') \leq g(s)$.

Proof:

We define $\mathcal{A}' = (S', I', T', F')$ with

- $S' = \mathcal{R} \times 2^S$;
- $I' = \{\langle g_0, \varnothing\rangle \mid g_0 \in \mathcal{R}, g_0(s) = \bot \text{ iff } s \notin I\}$;
- $T = \{(\langle g, \varnothing\rangle, \sigma, \langle g', P'\rangle) \mid g' \text{ covers } g, \text{ and } P' = \{s' \in S \mid g'(s') \text{ is even }\}\}$
  $\cup\ \{(\langle g, P\rangle, \sigma, \langle g', P'\rangle) \mid P \neq \varnothing, g' \text{ covers } g, \text{ and}$
  $P' = \{s' \in S \mid (s, \sigma, s') \in T, s \in P, g'(s') \text{ is even }\}\}$;
- $F = \mathcal{R} \times \{\varnothing\}$.

(Intuition: $\mathcal{A}'$ guesses the level rankings for the run DAG. The P component tracks the states whose corresponding vertices in the run DAG have even ranks. Paths that traverse such vertices should eventually reach a vertex with odd rank. The acceptance condition ensures that all paths visit a vertex with odd rank infinitely often.)

$\mathcal{L}(\mathcal{A}') \subseteq \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$:

- Let $\alpha \in \mathcal{L}(\mathcal{A}')$ and let $r' = (g_0, P_0), (g_1, P_1), \ldots$ be an accepting run of $\mathcal{A}'$ on α.
- Let $G = (V, E)$ be the run DAG of $\mathcal{A}$ on α.
- The function $f : \langle s, l\rangle \mapsto g_l(s), s \in S_l, l \in \omega$ is a ranking for G:
  - if $g_i(s)$ is odd then $s \notin F$;
  - for all $(\langle s, l\rangle, \langle s', l+1\rangle) \in E$, $g_{l+1}(s') \leq g_l(s)$.
- f is an odd ranking:
  - Assume otherwise. Then there exists a path $\langle s_0, l_0\rangle, \langle s_1, l_1\rangle, \langle s_2, l_2\rangle, \ldots$ in G such that for infinitely many $i \in \omega$, $f(\langle s_i, l_i\rangle)$ is even.
  - Hence, there exists an index $j \in \omega$, such that $f(\langle s_j, l_j\rangle)$ is even and, for all $k \geq 0$, $f(\langle s_{j+k}, l_{j+k}\rangle) = f(\langle s_j, l_j\rangle)$.
  - Since r' is accepting, $P_{j'} = \varnothing$ for infinitely many j'. Let j' be the smallest such index $\geq j$.
  - $P_{j'+1+k} \neq \varnothing$ for all $k \geq 0$.
  - Contradiction.

- Since there exists an odd ranking, $\alpha \notin \mathcal{L}(\mathcal{A})$.

$\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$:

- Let $\alpha \in \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$ and let $G = (V, E)$ be the run DAG of $\mathcal{A}$ on α.
- There exists an odd ranking f on G.
- There is a run $r' = (g_0, P_0), (g_1, P_1), \ldots$ of $\mathcal{A}'$ on α, where
$$g_l(s) = \begin{cases} f(\langle s, l \rangle) & \text{if } s \in S_l; \\ \bot & \text{otherwise;} \end{cases}$$
$P_0 = \varnothing$,
$$P_{l+1} = \begin{cases} \{s \in S \mid g_{l+1}(s) \text{ is even}\} & \text{if } P_l = \varnothing, \\ \{s' \in S \mid \exists s \in S_l \cap P_l \,.\, (\langle s, l \rangle, \langle s', l+1 \rangle) \in E, g_{l+1}(s') \text{ is even}\} & \text{otherwise.} \end{cases}$$
- r' is accepting. (Assume there is an index i such that $P_j \neq \varnothing$ for all $j \geq i$. Then there exists a path in G that visits an even rank infinitely often.)
- Hence, $\alpha \in \mathcal{L}(\mathcal{A}')$.

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