Bernd Finkbeiner Date: November 13, 2012

Automata, Games, and Verification: Lecture 5

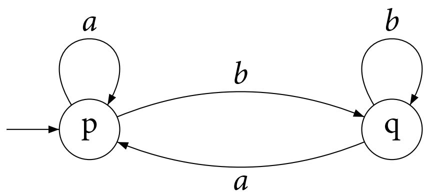
6 Muller Automata

Definition 1 *A* (nondeterministic) Muller automaton $\mathcal{A}$ *over alphabet* Σ *is a tuple* $(S, I, T, \mathcal{F})$*:*

- S, I, T *: defined as before*
- $\mathcal{F} \subseteq 2^S$ *: set of* accepting subsets*, called the* table.

Definition 2 *A run r of a Muller automaton is* accepting *iff* $In(r) \in \mathcal{F}$

Example:



- for $\mathcal{F} = \{\{q\}\}$: $\mathcal{L}(\mathcal{A}) = (a \cup b)^* b^\omega$
- for $\mathcal{F} = \{\{q\}, \{p, q\}\}$: $\mathcal{L}(\mathcal{A}) = (a^* b)^\omega$

Theorem 1 *For every (deterministic) Büchi automaton $\mathcal{A}$, there is (deterministic) Muller automaton $\mathcal{A}'$, such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

$S' = S, I' = I, T' = T$
$\mathcal{F}' = \{Q \subseteq S \mid Q \cap F \neq \emptyset\}$ ■

Theorem 2 *For every nondeterministic Muller automaton $\mathcal{A}$ there is a nondeterministic Büchi automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

- $\mathcal{F}' = \{F_1, \ldots, F_n\}$
- $S' = S \cup \bigcup_{i=1}^{n} \{i\} \times F_i \times 2^{F_i}$
- $I' = I$
- $T' = T$
  $\cup \ \{(s, \sigma, (i, s', \varnothing)) | 1 \leq i \leq n, (s, \sigma, s') \in T, s' \in F_i\}$
  $\cup \ \{((i, s, R), \sigma, (i', s', R')) \mid 1 \leq i \leq n, s, s' \in F_i, R, R' \subseteq F_i,$
  $(s, \sigma, s') \in T, R' = R \cup \{s\}$ if $R \neq F_i$ and $R' = \varnothing$ if $R = F_i\}$
- $F' = \bigcup_{i=1}^{n} \{i\} \times F_i \times \{F_i\}$

■

Theorem 3 *The languages recognizable by deterministic Muller automata are closed under Boolean operations (complementation, union, intersection).*

Proof:

- $\mathcal{L}(\mathcal{A}') = \Sigma^\omega \setminus \mathcal{L}(\mathcal{A})$:
  - $S' = S, I' = I, T' = T, \mathcal{F}' = 2^S \setminus \mathcal{F}$
- $\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$:
  - $S' = S_1 \times S_2, I' = I_1 \times I_2$,
  - $T' = \{((s_1, s_2), \sigma, (s_1', s_2')) \mid (s_1, \sigma, s_1') \in T_1, (s_2, \sigma, s_2') \in T_2\}$
  - $\mathcal{F}' = \{\Delta \subseteq S' | \ pr_1(\Delta) \in \mathcal{F}_1 \wedge pr_2(\Delta) \in \mathcal{F}_2\}$
    where $pr_i(Q) = \{p_i | (p_1, p_2, ..., p_i, ..., p_n) \in Q\}$
- $\mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2) = \Sigma^\omega \setminus ((\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}_1)) \cap (\Sigma^\omega \setminus \mathcal{L}(\mathcal{A}_2)))$.

■

Theorem 4 *A language $\mathcal{L}$ is recognizable by a deterministic Muller automaton iff $\mathcal{L}$ is a boolean combination of languages $\overrightarrow{W}$ where $W \subseteq \Sigma^*$ is regular.*

Proof:

($\Leftarrow$)

- If W is regular, then $\overrightarrow{W}$ is recognizable by a deterministic Büchi automaton;
- hence, $\overrightarrow{W}$ is recognizable by a deterministic Muller automaton;
- hence, the boolean combination $\mathcal{L}$ is recognizable by a deterministic Muller automaton.

($\Rightarrow$)

$\mathcal{A}$ accepts α with unique run r

iff for some $F \in \mathcal{F}$, $In(r) = F$

iff $\bigvee_{F\in\mathcal{F}} (\bigwedge_{s\in F} \alpha \in \overrightarrow{W_s} \wedge \bigwedge_{s\in S\setminus F} \alpha \notin \overrightarrow{W_s})$,
where $W_s = \mathcal{L}(\mathcal{A}_s)$ for the finite-word automaton $\mathcal{A}_s = (S, I, T, \{s\})$.

iff $\alpha \in \bigcup_{F\in\mathcal{F}} (\bigcap_{s\in F} \overrightarrow{W_s} \cap \bigcap_{s\in S\setminus F} (\Sigma^\omega \setminus \overrightarrow{W_s}))$.

■

7 McNaughton's Theorem

Theorem 5 (McNaughton's Theorem (1966)) *Every Büchi recognizable language is recognizable by a deterministic Muller automaton.*

Definition 3 *A Büchi automaton (S, I, T, F) is called* semi-deterministic *if $S = N \uplus D$ is a partition of S, $F \subseteq D$, $pr_3(T \cap (D \times \Sigma \times S)) \subseteq D$, and $(D, \{d\}, T \cap (D \times \Sigma \times D), F)$ is deterministic for every $d \in D$.*

Lemma 1 *For every Büchi automaton $\mathcal{A}$ there exists a semi-deterministic Büchi automaton $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Given $\mathcal{A} = (S, I, T, F)$, we construct $\mathcal{A}' = (S', I', T', F')$:

- $S' = S \uplus (2^S \times 2^S)$;
- $I' = I$;
- $$\begin{aligned} T' = T \;\cup\; &\{(s, \sigma, (\{s'\}, \varnothing)) \mid (s, \sigma, s') \in T\} \\ \cup &\{((L_1, L_2), \sigma, (L_1', L_2')) \mid L_1 \neq L_2 \\ &\quad L_1' = pr_3(T \cap L_1 \times \{\sigma\} \times S), \\ &\quad L_2' = pr_3(T \cap L_1 \times \{\sigma\} \times F) \cup pr_3(T \cap L_2 \times \{\sigma\} \times S)\} \\ \cup &\{((L, L), \sigma, (L_1', L_2')) \mid L_1' = pr_3(T \cap L \times \{\sigma\} \times S), \\ &\quad L_2' = pr_3(T \cap L \times \{\sigma\} \times F)\} \end{aligned}$$
- $F' = \{(L, L) \mid L \neq \varnothing)\}$

Example: