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Automata, Games, and Verification: Lecture 7

Lemma 1 *For every semi-deterministic Büchi automaton $\mathcal{A}$ there exists a deterministic Muller automaton $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Let $\mathcal{A} = (N \uplus D, I, T, F)$, $d = |D|$, and let D be ordered by $<$. We construct the DMA $(S', \{s'_0\}, T', \mathcal{F})$:

- $S' = 2^N \times \{0, \dots, 2d\} \to D \cup \{\llcorner\!\lrcorner\}$
- $s'_0 = (\{N \cap I\}, (d_1, d_2, \dots, d_n, \llcorner\!\lrcorner, \dots, \llcorner\!\lrcorner))$,
  where $d_i < d_{i+1}, \{d_1, \dots, d_n\} = D \cap I\}$.
- $T' = \{((N_1, f_1), \sigma, (N_2, f_2)) \mid N_2 = pr_3(T \cap N_1 \times \{\sigma\} \times N)$
  $D' = pr_3(T \cap N_1 \times \{\sigma\} \times D)$
  $g_1 : n \mapsto d_2 \in D \Leftrightarrow f_1 : n \mapsto d_1 \in D \wedge d_1 \to^\sigma d_2$

  g_2: insort the elements of D' in the empty slots of g_1 (using $<$)
  f_2: delete every recurrance (leaving an *empty* slot)
- $\mathcal{F} = \{F' \subseteq S' \mid \exists i \in 1, \dots, 2d$ s.t.

  $f(i) \neq \llcorner\!\lrcorner$ for all $(N', f) \in F'$ and

  $f(i) \in F$ for some $(N', f) \in F'\}$.

Proof:

$\underline{\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')}$:
If $\alpha \in \mathcal{L}(\mathcal{A})$, $\mathcal{A}$ has an accepting run $r = n_0 \dots n_{j-1} d_j d_{j+1} d_{j+2} \dots$
where $n_k \in N$ for $k < j$ and $d_k \in D$ for $k \geq j$.
Consider the run $r' = (N_0, f_0), (N_1, f_1), \dots$ of $\mathcal{A}'$ on α.

- $n_k \in N_k$ for all $k < j$,
- for all $k \geq j$, $d_k = f_k(i)$ for some $i \leq 2d$,
- these i's are non-increasing, and hence stabilize eventually.
- for this stable i,
  $f(i) \neq \llcorner\!\lrcorner$ for all $(N', f) \in In(r')$ and $f(i) \in F$ for some $(N', f) \in In(r')$.
- $In(r') \in \mathcal{F}$.

$\underline{\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})}$:
For $\alpha \in \mathcal{L}(\mathcal{A}')$, $\mathcal{A}'$ has an accepting run $r' = (N_0, f_0), (N_1, f_1), \dots$.

- We pick an i and an accepting set $F' \in \mathcal{F}$ s.t.
  $f(i) \neq \llcorner\!\lrcorner$ for all $(N', f) \in F'$ and $f(i) \in F$ for some $(N', f) \in F'$.

- We pick a $j \in \omega$ such that $f_n(i) \neq \sqcup$ for all $n > j$.
- There is a run $r = s_0 s_1 \ldots s_j f_{j+1}(i) f_{j+2}(i) f_{j+3}(i) \ldots$ of $\mathcal{A}$ for α.
- r is accepting.

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Theorem 1 (McNaughton's Theorem (1966)) *Every Büchi recognizable language is recognizable by a deterministic Muller automaton.*

8 LTL and QPTL

LTL: Linear-time temporal logic, QPTL: Quantified propositional temporal logic.

LTL: $a \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \psi \quad | \quad \mathrm{X}\varphi \quad | \quad \varphi \, \mathrm{U} \, \psi$
where $a \in AP$ and φ and ψ are LTL formulas. Additional derived operators: F and G.

QPTL: $\varphi \quad | \quad \psi \wedge \eta \quad | \quad \neg\psi \quad | \quad \exists p.\, \psi$
where φ is an LTL formula and ψ and η are QPTL formulas.

QPTL Semantics:

$$\alpha \models \exists q.\varphi \quad \text{iff} \quad \begin{array}{l} \text{there is an } \alpha' \text{ with} \\ \alpha'(j) \cap (AP \setminus \{q\}) = \alpha(j) \cap (AP \setminus \{q\}) \text{ for all } j \in \omega, \\ \text{s.t. } \alpha' \models \varphi. \end{array}$$

There are Büchi recognizable languages that are not LTL-definable.
Example: $(\varnothing\varnothing)^*\{p\}^\omega$

Definition 1 *A language $L \subseteq \Sigma^\omega$ is* non-counting *iff*
$\exists n_0 \in \omega \,.\, \forall n \geq n_0 \,.\, \forall u, v \in \Sigma^*, \gamma \in \Sigma^\omega \,.$
$uv^n\gamma \in L \;\Leftrightarrow\; uv^{n+1}\gamma \in L$

Example: $L = (\varnothing\varnothing)^*\{p\}^\omega$ is counting. For every $\varnothing^n\{p\}^\omega \in L$, $\varnothing^{n+1}\{p\}^\omega \notin L$. ✚

Theorem 2 *For every LTL-formula φ, $\mathcal{L}(\varphi)$ is non-counting.*

Proof:

Structural induction on φ:

- $\varphi = p$: choose $n_0 = 1$.
- $\varphi = \varphi_1 \wedge \varphi_2$: By IH, φ_1 defines non-counting language with threshold $n_0' \in \omega$, φ_2 with n_0''; choose $n_0 = \max(n_0', n_0'')$;
- $\varphi = \neg\varphi_1$: choose $n_0 = n_0'$.
- $\varphi = \mathrm{X}\varphi_1$: choose $n_0 = n_0' + 1$.
  - We show for $n \geq n_0$: $uv^n\gamma \models \mathrm{X}\varphi \;\Leftrightarrow\; uv^{n+1}\gamma \models \mathrm{X}\varphi$.

– Case $u \neq \varepsilon$, i.e., $u = au'$ for some $a \in \Sigma, u' \in \Sigma^*$:

$$\begin{aligned} & au'v^n\gamma \vDash X\varphi \\ \text{iff } & u'v^n\gamma \vDash \varphi \\ \text{iff } & u'v^{n+1}\gamma \vDash \varphi \qquad \text{(IH)} \\ \text{iff } & au'v^{n+1}\gamma \vDash X\varphi. \end{aligned}$$

– Case $u = \varepsilon, v = av'$ for some $a \in \Sigma, v' \in \Sigma^*$:

$$\begin{aligned} & (av')^n\gamma \vDash X\varphi \\ \text{iff } & (av')(av')^{n-1}\gamma \vDash X\varphi \\ \text{iff } & v'(av')^{n-1}\gamma \vDash \varphi \\ \text{iff } & v'(av')^n\gamma \vDash \varphi \qquad \text{(IH)} \\ \text{iff } & (av')^{n+1}\gamma \vDash X\varphi. \end{aligned}$$

- $\varphi = \varphi_1 \,\mathcal{U}\, \varphi_2$: choose $n_0 = \max(n_0', n_0'') + 1$.
  Claim: for $n \geq n_0$: $uv^n\gamma \vDash \varphi_1 \,\mathcal{U}\, \varphi_2 \;\Rightarrow\; uv^{n+1}\gamma \vDash \varphi_1 \,\mathcal{U}\, \varphi_2$.
  – $uv^n\gamma \vDash \varphi_1 \,\mathcal{U}\, \varphi_2 \;\Rightarrow\; \exists j \,.\, uv^n\gamma[j..] \vDash \varphi_2$ and $\forall i < j \,.\, uv^n\gamma[i..] \vDash \varphi_1$.
  – Let j be the least such index.
  – Case $j \leq |u|$:
  by IH, $uv^{n+1}\gamma[j..] \vDash \varphi_2$ and for all $i < j \,.\, uv^{n+1}\gamma[i..] \vDash \varphi_1$;
  – Case $j > |u|$:
  $uv^{n+1}\gamma[j + |v|..] \vDash \varphi_2$ (because $uv^{n+1}\gamma$ has the same suffix from position $j + |v|$ as uv^{n+1} from position j);
  for all $|u| + |v| \leq i < j + |v| \,.\, uv^{n+1}\gamma[i..] \vDash \varphi_1$ (again, because the suffix is the same);
  By (IH), for all $i < |u| + |v|, i < j \,.\, uvv^n\gamma[i..] \vDash \varphi_1$, because $uvv^{n-1}\gamma[i..] \vDash \varphi_1$.

  Claim: for $n \geq n_0$: $uv^{n+1}\gamma \vDash \varphi_1 \,\mathcal{U}\, \varphi_2 \;\Rightarrow\; uv^n\gamma \vDash \varphi_1 \,\mathcal{U}\, \varphi_2$
  – $uv^{n+1}\gamma \vDash \varphi_1 \,\mathcal{U}\, \varphi_2 \;\Rightarrow\; \exists j \,.\, uv^{n+1}\gamma[j..] \vDash \varphi_2$ and $\forall i < j \,.\, uv^{n+1}\gamma[i..] \vDash \varphi_1$.
  – Case $j \leq |u| + |v|$:
  by IH, $uvv^{n-1}[j..] \vDash \varphi_2$ and for all $i < j \,.\, uvv^{n-1}[i..] \vDash \varphi_1$;
  – Case $j > |u| + |v|$:
  $uv^n\gamma[j - |v|..] \vDash \varphi_2$;
  for all $|u| + |v| \leq i < j \,.\, uv^n\gamma[i..] \vDash \varphi_1$;
  By (IH), for all $i < |u| + |v| \,.\, uvv^{n-1}\gamma[i..] \vDash \varphi_1$, because $uvv^n\gamma[i..] \vDash \varphi_1$.

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Example: $L = (\varnothing\varnothing)^*\{p\}^\omega$ is QPTL-definable:
$\exists q. \,(q \wedge G(q \leftrightarrow \neg Xq) \wedge G(p \rightarrow Xp) \wedge G(Xp \rightarrow (p \vee \neg q)))$ ✚

Theorem 3 *For every Büchi automaton $\mathcal{A}$ over $\Sigma = 2^{AP}$ there exists a QPTL formula φ such that $\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A})$.*

Proof:

Let $S = \{s_1, s_2, \ldots, s_n\}$ and $AP' = AP \cup \{at_{s_1}, \ldots, at_{s_n}\}$.

$$
\begin{aligned}
\varphi \ := \ \exists at_{s_1}, \ldots, at_{s_n} \quad . \quad & \bigvee_{s \in I} at_s \\
\wedge \quad & \mathrm{G}\left(\bigvee_{(s_i, A, s_j) \in T} at_{s_i} \wedge \mathrm{X} at_{s_j} \wedge \left(\bigwedge_{p \in A} p \right) \wedge \left(\bigwedge_{p \in AP \setminus A} \neg p \right) \right) \\
\wedge \quad & \mathrm{G}\left(\bigvee_{i=1}^{n} \bigwedge_{j \neq i} \neg (at_{s_i} \wedge at_{s_j}) \right) \\
\wedge \quad & \mathrm{GF} \bigvee_{s_i \in F} at_{s_i}
\end{aligned}
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9 Monadic Second-Order Theory of One Successor (S1S)

Syntax:

- first-order variable set $V_1 = \{x, y, \ldots\}$
- second-order variable set $V_2 = \{X, Y, \ldots\}$
- Terms t:
$$t ::= 0 \mid x \mid S(t)$$
- Formulas φ:
$$\varphi ::= t \in X \mid t_1 = t_2 \mid \neg\varphi \mid \varphi_0 \vee \varphi_1 \mid \exists x.\varphi \mid \exists X.\varphi$$

Abbreviations:

- $\forall X.\, \varphi \ := \ \neg\exists X.\, \neg\varphi$;
- $x \notin Y \ := \ \neg(x \in Y)$;
- $x \neq y \ := \ \neg(x = y)$.

Semantics:

- first-order valuation $\sigma_1 : V_1 \to \omega$
- second-order valuation $\sigma_2 : V_2 \to 2^\omega$

Semantics of terms:

- $[0]_{\sigma_1} = 0$
- $[x]_{\sigma_1} = \sigma_1(x)$

- $[S(t)_{\sigma_1}] = [t]_{\sigma_1} + 1$

Semantics of formulas:

- $\sigma_1, \sigma_2 \vDash t \in X$ iff $[t]_{\sigma_1} \in \sigma_2(X)$
- $\sigma_1, \sigma_2 \vDash t_1 = t_2$ iff $[t_1]_{\sigma_1} = [t_2]_{\sigma_1}$
- $\sigma_1, \sigma_2 \vDash \neg\psi$ iff $\sigma_1, \sigma_2 \nvDash \psi$
- $\sigma_1, \sigma_2 \vDash \psi_0 \vee \psi_1$ iff $\sigma_1, \sigma_2 \vDash \psi_0$ or $\sigma_1, \sigma_2 \vDash \psi_1$
- $\sigma_1, \sigma_2 \vDash \exists x.\ \varphi$ iff there is an $a \in \omega$ s.t.

$$\sigma_1'(y) = \begin{cases} \sigma_1(y) \text{ if } y \neq x \\ a \text{ otherwise} \end{cases}$$

  and $\sigma_1', \sigma_2 \vDash \varphi$.

- $\sigma_1, \sigma_2 \vDash \exists X.\ \varphi$ iff there is an $A \subseteq \omega$ s.t.

$$\sigma_2'(Y) = \begin{cases} \sigma_2(Y) \text{ if } Y \neq X \\ A \text{ otherwise} \end{cases}$$

  and $\sigma_1, \sigma_2' \vDash \varphi$

Example:

$$\begin{array}{lcl}
X \subseteq Y & :\equiv & \forall z.\ (z \in X \rightarrow z \in Y); \\
X = Y & :\equiv & X \subseteq Y \wedge Y \subseteq X; \\
\mathit{Suff}(X) & :\equiv & \forall y.\ (y \in X \rightarrow S(y) \in X); \\
x \leq y & :\equiv & \forall Z.\ (x \in Z \wedge \mathit{Suff}(Z)) \rightarrow y \in Z; \\
\mathit{Fin}(X) & :\equiv & \exists Y.\ (X \subseteq Y \wedge (\exists z.\ z \notin Y) \wedge (\forall z.\ (z \notin Y \rightarrow S(z) \notin Y)));
\end{array}$$

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