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Automata, Games, and Verification: Lecture 9

Theorem 1 *For every LTL formula φ, there is an alternating Büchi automaton $\mathcal{A}_\varphi$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\varphi)$*

Proof:

- $S = \text{closure}(\varphi) := \{\psi, \neg\psi \mid \psi \text{ is subformula of } \varphi\}$;
- $s_0 = \varphi$;
- $\delta(p, a) = \mathit{true}$ if $p \in a$, false if $p \notin a$;
  $\delta(\neg p, a) = \mathit{false}$ if $p \in a$, true if $p \notin a$;
  $\delta(\mathit{true}, a) = \mathit{true}$;
  $\delta(\mathit{false}, a) = \mathit{false}$;
- $\delta(\psi_1 \wedge \psi_2, a) = \delta(\psi_1, a) \wedge \delta(\psi_2, a)$;
- $\delta(\psi_1 \vee \psi_2, a) = \delta(\psi_1, a) \vee \delta(\psi_2, a)$;
- $\delta(\mathrm{X}\,\psi, a) = \psi$;
- $\delta(\psi_1 \,\mathcal{U}\, \psi_2, a) = \delta(\psi_2, a) \vee (\delta(\psi_1, a) \wedge \psi_1 \,\mathcal{U}\, \psi_2)$;
- $\delta(\neg\psi, a) = \overline{\delta(\psi, a)}$;
- $\overline{\psi} = \neg\psi$ for $\psi \in S$;
- $\overline{\neg\psi} = \psi$ for $\psi \in S$;
- $\overline{\alpha \wedge \beta} = \overline{\alpha} \vee \overline{\beta}$;
- $\overline{\alpha \vee \beta} = \overline{\alpha} \wedge \overline{\beta}$;
- $\overline{\mathit{true}} = \mathit{false}$;
- $\overline{\mathit{false}} = \mathit{true}$;
- $F = \{\neg(\psi_1 \,\mathcal{U}\, \psi_2) \in \text{closure}(\varphi)\}$

For a subformula ψ of φ let $\mathcal{A}_\varphi^\psi$ be the automaton A_φ with initial state ψ.
Claim: $\mathcal{L}(\mathcal{A}_\varphi^\psi) = \mathcal{L}(\psi)$. Proof by structural induction. ∎

Definition 1 *Two nodes $x_1, x_2 \in T$ in a run tree (T, r) are* similar *if $|x_1| = |x_2|$ and $r(x_1) = r(x_2)$.*

Definition 2 *A run tree (T, r) is* memoryless *if for all similar nodes x_1 and x_2 and for all $y \in D^*$ we have that ($x_1 \cdot y \in T$ iff $x_2 \cdot y \in T$) and $r(x_1 \cdot y) = r(x_2 \cdot y)$.*

Theorem 2 *If an alternating Büchi Automaton $\mathcal{A}$ accepts a word α, then there exists a memoryless accepting run of $\mathcal{A}$ on α.*

Proof:

- Let (T, r) be an accepting run tree on α with directions D.
- We define $\gamma : T \to \omega$ (measures the number of steps since the last visit to F):
  - $\gamma(\varepsilon) = 0$
  - $\gamma(n \cdot d) = \begin{cases} \gamma(n) + 1 & \text{if } r(n) \notin F; \\ 0 & \text{otherwise;} \end{cases}$
- We define $\Delta : S \times \omega \to T$:
  $\Delta(s, n)$ = leftmost $y \in T$ with $|y| = n, r(y) = s$ and $(\forall z \in T, |z| = n \wedge r(z) = s \Rightarrow \gamma(z) \leq \gamma(y))$.
- We define (T', r'):
  - $\varepsilon \in T', r'(\varepsilon) = r(\varepsilon)$;
  - for $n \in T', d \in D$,
    $n \cdot d \in T'$ iff $\Delta(r'(n), |n|) \cdot d \in T$;
    $r'(n \cdot d) = r(\Delta(r'(n), |n|) \cdot d)$

Claim 1: (T', r') is a run of $\mathcal{A}$ on α.

- $r'(\varepsilon) = r(\varepsilon) = s_0$
- For $n \in T'$, let $q_n = \Delta(r'(n), |n|)$.
- For every $n \in T'$, $\{r(q_n \cdot d) \mid d \in D, q_n \cdot d \in T\} \models \delta(r(q_n), \alpha(|q_n|))$
  and therefore $\{r'(n \cdot d) \mid d \in D, n \cdot d \in T'\} \models \delta(r'(n), \alpha(|n|))$.

Claim 2: If (T, r) is accepting, then so is (T', r'). Proof by contradiction:

- **Claim 2.1** : For every $n \in T'$, $\gamma(n) \leq \gamma(\Delta(r'(n), |n|))$. Proof by induction on the length of n:
  - for $n = \varepsilon$, $\gamma(n) = 0$
  - for $n = n' \cdot d$ (where $d \in D$),
    * if $r(n') \in F$, then $\gamma(n) = 0$
    * if $r(n') \notin F$, then

$$
\begin{aligned}
&\gamma(\Delta(r'(n' \cdot d), |n' \cdot d|)) \\
\geq \quad & (\Delta \text{ definition}) \\
&\gamma(\Delta(r'(n'), |n'|) \cdot d) \\
= \quad & (\gamma \text{ definition}) \\
&1 + \gamma(\Delta(r'(n'), |n'|)) \\
\geq \quad & (\text{induction hypothesis}) \\
&1 + \gamma(n') \\
= \quad & (\gamma \text{ definition }) \\
&\gamma(n' \cdot d)
\end{aligned}
$$

- Suppose (T', r') is not accepting, then there is an infinite branch $\pi : n_0, n_1, n_2, \ldots \in T'$ and $\exists k \in \omega$ such that $\forall j \geq k : r'(b_j) \notin F$.

- Let $m_i = \Delta(r'(n_i), |n_i|)$ for $i \geq k$.
- We have,
  $$\begin{array}{ccccc} \gamma(n_k) & < & \gamma(n_{k+1}) & < & \dots \\ |\!\wedge & & |\!\wedge & & \\ \gamma(m_k) & < & \gamma(m_{k+1}) & < & \dots \end{array}$$
  So, for any $k' > k$, $\gamma(m_k) \geq k' - k$.
  Since T is finitely branching, there must be a branch with an infinite suffix of non-F labeled positions. This contradicts our assumption that (T, r) is accepting.

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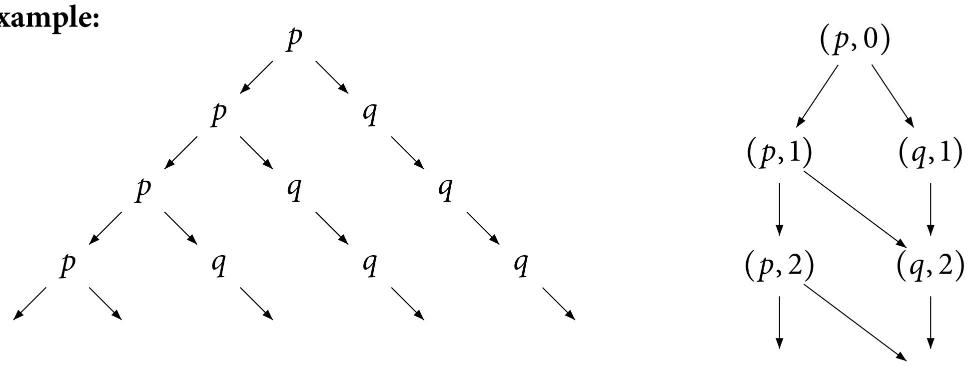
Definition 3 *A* run DAG *of an alternating Büchi Automaton $\mathcal{A}$ on word α is a DAG (V, E), where*

- $V \subseteq S \times \omega$
- $E \subseteq \bigcup_{i \in \omega} (S \times \{i\}) \times (S \times \{i+1\})$;
- $(s_0, 0) \in V$
- $\forall (s, i) \in V \,.\, \exists Y \subseteq S$ *s.t.*
  $Y \models \delta(s, \alpha(i))$, $Y \times \{i+1\} \subseteq V$ *and* $\{(s, i)\} \times (Y \times \{i+1\}) \subseteq E$.

Example:



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Notation: Level $((V, E), i) = \{s \in S \mid (s, i) \in V\}$

Definition 4 *A run DAG is* accepting *if every infinite path has infinitely many visits to $F \times \omega$.*

Corollary 1 *A word α is accepted by an alternating Büchi automaton $\mathcal{A}$ iff $\mathcal{A}$ has an accepting run DAG on α.*

Theorem 3 (Miyano and Hayashi, 1984) *For every alternating Büchi automaton $\mathcal{A}$, there exists a nondeterministic Büchi automaton $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.*

Proof:

- $S' = 2^S \times 2^S$;
- $I' = \{(\{s_0\}, \varnothing)\}$;
- $F' = \{(X, \varnothing) \mid X \subseteq S\}$;

- $T' = \{((X,\varnothing),\sigma,(X',X'-F)) \mid X' \vDash \bigwedge_{s\in X} \delta(s,\sigma)\}$
  $\cup \{((x,W),\sigma,(X',W' \setminus F)) \mid W \neq \varnothing, W' \subseteq X', X' \vDash \bigwedge_{s\in X} \delta(s,\sigma),$
  $W' \vDash \bigwedge_{s\in W} \delta(s,\sigma)\}$.

$\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$**:**

- Let $\alpha \in L(\mathcal{A}')$ with accepting run

$$r' : (X_0, W_0)(X_1, W_1)(X_2, W_2)\ldots$$

  where $W_0 = \varnothing, X_0 = \{s_0\}$.
- We construct the run DAG (V,E) for $\mathcal{A}$ on α:
  - $V = \bigcup_{i\in\omega} X_i \times \{i\}$;
  - $E = \bigcup_{i\in\omega} \quad (\bigcup_{x\in X_i \setminus W_i} \{(x,i)\} \times (X_{i+1} \times \{i+1\}))$
    $\cup \; (\bigcup_{x\in W_i} \quad \{(x,i)\} \times ((X_{i+1} \cap (F \cup W_{i+1})) \times \{i+1\}))$.
- (V,E) is an accepting run DAG:
  - $(s_0, 0) \in V$;
  - for $(x,i) \in V$:
    * if $x \in X_i \setminus W_i$, $X_{i+1} \vDash \delta(x,\alpha(i))$;
    * if $x \in W_i$, $X_{i+1} \cap (F \cup W_{i+1}) \vDash \delta(x,\alpha(i))$.
  - Every path through the run DAG visits F infinitely often (otherwise $W_i = \varnothing$ only for finitely many i).

$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$**:**

- Let $\alpha \in L(A')$ and (V,E) an accepting run DAG of $\mathcal{A}$ on α.
- We construct a run

$$r' : (X_0, W_0)(X_1, W_1)(X_2, W_2)\ldots$$

  on $\mathcal{A}'$ as follows:
  - $X_0 = \{s_0\}$, $W_0 = \varnothing$;
  - for $i > 0$, $X_i = \mathit{Level}((V,E),i)$
    * if $W_i = \varnothing$ then $W_{i+1} = X_{i+1} \setminus F$,
    * otherwise,
      $W_{i+1} := \{y' \in S \setminus F \mid \exists (y,i) \in V, ((y,i),(y',i+1)) \in E, y \in W_i\}$.
- r' is an accepting run:
  - starts with $(\{s_0\},\varnothing)$
  - obeys T':
    * for $x \in X_i \setminus W_i$, $X_{i+1} \vDash \delta(x,\alpha(i))$;
    * for $x \in W_i$, $X_{i+1} \cap (F \cup W_{i+1}) \vDash \delta(x,\alpha(i))$.
  - r' is accepting (otherwise there exists a path in (V,E) that is not accepting).

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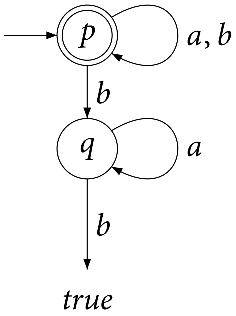
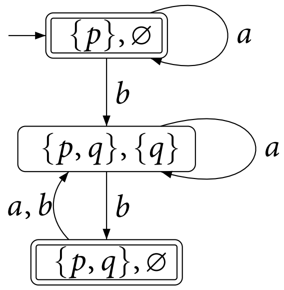
Example: We translate the following *universal* automaton (all branchings are conjunctions) into an equivalent nondeterministic automaton:





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Corollary 2 *A language is ω-regular iff it is recognizable by an alternating Büchi automaton.*

Proof:

Translation from nondeterministic Büchi automaton $(S, \{s_0\}, T, F)$ to alternating Büchi automaton (S, s_0, δ, F) with

- $\delta(s, \sigma) = \bigvee_{s' \in pr_3(T \cap \{s\} \times \{\sigma\} \times S)} s'$ for all $s \in S$

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Comment: Acceptance of a word α by an alternating Büchi automaton can also be characterized as a game:

- Positions of Player 0: $V_0 = S \times \omega$;
- Positions of Player 1: $V_1 = 2^S \times \omega$;
- Edges: $\{((s, i), (X, i)) \mid X \models \delta(s, \alpha(i))\}$
  $\cup \{((X, i), (s, i+1)) \mid s \in X\}$

Player 0 wins a play iff $F \times \omega$ is visited infinitely often.

The word α is accepted iff Player 0 has a strategy to win the game from position $(s_0, 0)$. ✚