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Problem Set 2
discussion session: 6th of November 2012

Automata, Games, and Verification

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1. **ω-Regular Expressions** (Group G01, discussion session 12:00 with Hazem Torfah)

   Represent each of the following ω-languages over the alphabet $\{a, b\}$ as a finite union of languages $V \cdot W^\omega$, where each V and W is recognizable by an automaton on finite words:

   a) $L_1 = \{\alpha \mid$ the maximal substrings of α consisting of only a's have even length$\}$

   b) $L_2 = \{\alpha \mid$ each a is preceeded by a b in $\alpha\}$

   c) $L_3 = \{\alpha \mid \alpha$ has no occurrence of $bab\}$

2. **Deterministic Büchi Automata** (Group G06, discussion session 12:00 with Felix Klein)

   Let Σ be an alphabet of the form $\Sigma = \Sigma_1 \times \Sigma_2 = \{(a, b) \mid a \in \Sigma_1, b \in \Sigma_2\}$, where Σ_1 and Σ_2 are also alphabets. Let L be a language over the alphabet Σ. We define the *projections* $pr_1(L)$ and $pr_2(L)$ as follows:

   $$pr_1(L) = \{\ u_0u_1u_2 \ldots \in \Sigma_1^\omega \mid \exists\ v_0v_1v_2 \ldots \in \Sigma_2^\omega \text{ s.t. } (u_0, v_0)(u_1, v_1)(u_2, v_2) \ldots \in L\ \}$$
   $$pr_2(L) = \{\ v_0v_1v_2 \ldots \in \Sigma_2^\omega \mid \exists\ u_0u_1u_2 \ldots \in \Sigma_1^\omega \text{ s.t. } (u_0, v_0)(u_1, v_1)(u_2, v_2) \ldots \in L\ \}$$

   Prove or give a counterexample to the following statements:

   a) Deterministic Büchi automata are closed under $\cap$.

   b) Deterministic Büchi automata are closed under $\cup$.

   c) Deterministic Büchi automata are closed under pr_1.

3. **Limit Operation** (Group G11, discussion session 12:20 with Felix Klein)

   a) Let $V, W \subseteq \Sigma^*$ be two regular languages. Prove or give a counterexample to the following equation:

   $$\overrightarrow{(V \cdot W)} = V \cdot \overrightarrow{W}$$

   b) Let $\mathcal{A} = (S, \{s_0\}, T, F)$ be an automaton on finite words. Let $L_* = \mathcal{L}(\mathcal{A})$ be the language of $\mathcal{A}$ and let L_ω be the language of $\mathcal{A}$ when it is regarded as a Büchi automaton. Prove or give a counterexample for the following equation:

   $$L_\omega = \overrightarrow{L_*}$$

4. **Universal Projection** (Group G15, discussion session 12:40 with Hazem Torfah)

   We define the following "universal flavor" of projection (for $i \in \{1,2\}$ and $L \subseteq (\Sigma_1 \times \Sigma_2)^\omega$):

$$rp_i(L) = \{\alpha' \in \Sigma_i^\omega \mid \text{ for every } \alpha \in (\Sigma_1 \times \Sigma_2)^\omega \,.\, pr_i(\alpha) = \alpha' \Rightarrow \alpha \in L\}$$

   Show that if L is Büchi recognizable, then so is $rp_1(L)$.

   Hint: You might want to use that the complement of any Büchi recognizable language is Büchi recognizable.

5. **Projection and Büchi Recognizable Languages (Challenge)**

   (a) Prove that the projections $pr_1(L)$ and $pr_2(L)$ of a Büchi recognizable language L on the alphabet $\Sigma_1 \times \Sigma_2$ are Büchi recognizable.

   (b) Prove that the converse of (a) is false: Construct a non-Büchi recognizable ω-language L such that both $pr_1(L)$ and $pr_2(L)$ are Büchi recognizable.

   Hint: The language $L' = \{a^n b^n \mid n = 1, 2, 3, \ldots\}^\omega$ over the alphabet $\{a, b\}$ is not Büchi-recognizable. Some variation of L' is useful to construct the required language L.