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Problem Set 8
Discussions: June 17th, 2015

Automata, Games, and Verification

Please send a mail to `agv15@react.uni-saarland.de` if you can't make it to the discussion session.

1. **S1S and LTL** (presented by Group 02)

   Decide for each of the languages over $2^{\{p,q\}}$ described below if they can be defined in S1S and/or LTL. Justify your answer in each case by either providing a formula or an argument why the language is not definable.

   a) $L_1 = \{\alpha \mid p \in \alpha(0),\, p \notin \alpha(i) \text{ for all } i \geq 1\}$
   b) $L_2 = \{\alpha \mid p \in \alpha(i) \text{ for exactly two different } i \in \mathbb{N}\}$
   c) $L_3 = \{\alpha \mid |\{i \in \mathbb{N} \mid p \in \alpha(i)\}| \text{ is finite and even}\}$
   d) $L_4 = \{\alpha \mid |\{i \in \mathbb{N} \mid p \in \alpha(i)\}| \text{ and } |\{i \in \mathbb{N} \mid q \in \alpha(i)\}| \text{ are finite and equal}\}$

2. **LTL, QPTL & S1S** (presented by Group 12)

   Let $AP = \{q, p, r\}$. Given some word $\alpha = \alpha_0\alpha_1\alpha_2 \ldots \in (2^{AP})^\omega$, for every $a \in AP$, we denote by $\alpha|_a$ the word $(\alpha_0 \cap \{a\})(\alpha_1 \cap \{a\})(\alpha_2 \cap \{a\}) \ldots$.

   Given some finite word $w = w_0w_1 \ldots w_n$, we define $f\colon 2^{AP} \to \mathbb{N}$ to denote the number represented by w in binary (with the least significant bit first), where we treat the letter $\emptyset$ as 0 and every other letter in 2^{AP} as 1, i.e., $f(\epsilon) = 0$ and

   $$f(w_0w_1 \ldots w_n) = \begin{cases} f(w[1,n])) \cdot 2 & \text{if } w_0 = \emptyset \\ f(w[1,n])) \cdot 2 + 1 & \text{if } w_0 \neq \emptyset \end{cases}$$

   Represent the following language L as LTL, QPTL and S1S formulas.

   $$L = \{\alpha \in (2^{AP})^\omega \mid \forall j \in \mathbb{N}.\ f(\alpha|_r[0,j]) = f(\alpha|_p[0,j]) + f(\alpha|_q[0,j])\}$$

3. **Projecting S1S & LTL** (presented by Group 15)

   Let $L \subseteq (2^{AP})^\omega$ be an LTL-definable language and let $AP' \subsetneq AP$ be a strict subset of AP. Prove or give a counter example to the following statements:

   a) The (weak) projection $L_w = \{\alpha' \in (2^{AP'})^\omega \mid \exists \alpha \in L.\ \forall i \in \mathbb{N}.\ \alpha'(i) = \alpha(i) \cap AP'\}$ of L is LTL-definable.
   b) The (weak) projection L_w of L is S1S-definable.
   c) The strong projection $L_s = \{\alpha' \in (2^{AP'})^\omega \mid \forall \alpha \in (2^{AP})^\omega.\ (\forall i \in \mathbb{N}.\ \alpha'(i) = \alpha(i) \cap AP') \to \alpha \in L\}$ of L is LTL-definable.
   d) The strong projection L_s of L is S1S-definable.

4. **S1S Characterization** (presented by Group 06)

   Prove or give a counter-example to the following: Every S1S definable language L is definable by an S1S formula φ which is in prenex normal form, i.e., the $\forall$ and $\exists$ quantifiers are situated at the front of the formula, and where second-order quantification is restricted to $\exists$.

5. **Presburger Arithmetic (Challenge)**

Presburger arithmetic is a fragment of natural number arithmetic involving constants, addition, inequalities, and quantification. Since its validity problem is decidable, decision procedures for it can be used in theorem provers to automatically determine whether an arithmetic property is a theorem. The syntax of Presburger arithmetic is defined as follows:

Terms $\quad t \ ::= \ 0 \mid 1 \mid v \mid t_1 + t_2,$

where v is a variable $v \in V$ chosen from a set of variables V.

Formulas $\quad \varphi \ ::= \ t_1 \geq t_2 \mid \neg\varphi_1 \mid \varphi_1 \vee \varphi_2 \mid \exists v.\, \varphi_1$

The semantics are defined in a straightforward way relative to a valuation $\sigma\colon V \to \mathbb{N}$ of variables. Using a decision procedure for S1S, we develop a decision procedure for Presburger Arithmetic satisfiability:

a) Are Presburger formulas a special case of S1S formulas (disregarding second-order variables)?

b) Describe a syntactic transformation T which provides an S1S formula given any Presburger formula φ which is satisfiable if and only if the original formula φ is satisfiable.