Chapter 6

Arrays

DECISION PROCEDURES
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6.1 The modelclass

6.1.1 Definition
The SIGNATURE OF ARRAYS Σ_{array} contains the following symbols:

- The sorts elem for elements, index for indices, and array for arrays.
- The function symbol read, of arity $\mathsf{array} \times \mathsf{index} \to \mathsf{elem}$.
- The function symbol write, of arity $\mathsf{array} \times \mathsf{index} \times \mathsf{elem} \to \mathsf{array}$.

6.1.2 Definition
A STANDARD array-STRUCTURE is any Σ_{array}-structure $\mathcal{A}$ satisfying the following properties:

1. $A_{\mathsf{array}} = (A_{\mathsf{elem}})^{A_{\mathsf{index}}}$.
2. $\mathsf{read}^{\mathcal{A}}(a, i) = a(i)$, for all $a \in A_{\mathsf{array}}$ and $i \in A_{\mathsf{index}}$.
3. $\mathsf{write}^{\mathcal{A}}(a, i, e) = a_{i \mapsto e}$, for all $a \in A_{\mathsf{array}}$, $i \in A_{\mathsf{index}}$, and $e \in A_{\mathsf{elem}}$.

6.1.3 Definition
The MODELCLASS OF ARRAYS is the pair $M_{\mathsf{array}} = (\Sigma_{\mathsf{array}}, \mathbf{A})$, where $\mathbf{A}$ is the class of all Σ_{array}-structures that are isomorphic to standard array-structures.

6.2 Equality reduction

6.2.1 Algorithm (IS-SATISFIABLE-ARRAY)
Input: A finite set Γ of flat Σ_{array}-literals

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Output: `satisfiable` if Γ is satisfiable; `unsatisfiable` otherwise

```
function IS-SATISFIABLE-SETS(Γ)
    Δ ← ARRAYS-TO-EQUALITY(Γ)
    return IS-SATISFIABLE-EQUALITY(Δ)
end function

function ARRAYS-TO-EQUALITY(Γ)
    X ← vars(Γ)
    For each literal of the form a ≉_array b in Γ, generate a fresh index-
    variable w_{a,b}. Let W be the set of freshly generated variables.
    Replace each formula of the form a ≉_array b in Γ with the formula

        read(a, w_{a,b}) ≉ read(b, w_{a,b})

    Replace each formula of the form a ≈ write(b, i, e) in Γ with the formula

        ⋀_{j ∈ X_index ∪ W} if i ≈ j then read(a, j) ≈ e else read(a, j) ≈ read(b, j)

    return Γ
end function
```

6.2.2 Proposition

In algorithm IS-SATISFIABLE-ARRAYS*, let Δ be the output returned by the call to* ARRAYS-TO-EQUALITY*(Γ). Then the following are equivalent:*

1. *Γ is M_{array}-satisfiable.*
2. *Δ is $M_{\approx}^{\Omega}$-satisfiable, where $\Omega^{\mathrm{S}} = \{\mathsf{elem}, \mathsf{index}, \mathsf{array}\}$, $\Omega^{\mathrm{C}} = \Omega^{\mathrm{F}} = \{\mathsf{read}\}$, and $\Omega^{\mathrm{P}} = \varnothing$.*

PROOF. ($1 \implies 2$). Immediate.

($1 \implies 2$). Let $\mathcal{B}$ be an Ω-interpretation over $X \cup W$ satisfying Δ. Then it is easy to check that Γ is satisfied by the M_{array}-interpretation $\mathcal{A}$ over X constructed as follows. We first let

$$A_{\mathsf{elem}} = B_{\mathsf{elem}} \,,$$
$$A_{\mathsf{index}} = B_{\mathsf{index}} \,,$$

and

$$e^{\mathcal{A}} = e^{\mathcal{B}} \,, \qquad \text{for each } e \in X_{\mathsf{elem}} \,,$$
$$i^{\mathcal{A}} = i^{\mathcal{B}} \,, \qquad \text{for each } i \in X_{\mathsf{index}} \,.$$

Moreover, for each $a \in X_{\mathsf{array}}$ and $i \in A_{\mathsf{index}}$, we let

$$a^{\mathcal{A}}(i) = \begin{cases} \mathsf{read}^{\mathcal{B}}(a^{\mathcal{B}}, i)\,, & \text{if } i \in (X_{\mathsf{index}} \cup W)^{\mathcal{B}}\,, \\ e_0\,, & \text{otherwise}\,, \end{cases}$$

where e_0 is an arbitrarily fixed element in A_{elem}.

6.2.3 Proposition
Algorithm IS-SATISFIABLE-ARRAYS *is correct.*

PROOF. Termination is obvious. Partial correctness follows by Proposition 6.2.2.