Chapter 7

Sets

Decision Procedures
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7.1 The modelclass

7.1.1 Definition
The signature of sets Σ_{set} contains the following symbols:

- The sorts elem for elements and set for sets of elements.
- The constant symbol $\varnothing$ (*empty set*), of sort set.
- The function symbol $\{\cdot\}$ (*singleton set*), of arity $\mathsf{elem} \to \mathsf{set}$.
- The function symbol $\cup$ (*union*), of arity $\mathsf{set} \times \mathsf{set} \to \mathsf{set}$.
- The function symbol $\cap$ (*intersection*), of arity $\mathsf{set} \times \mathsf{set} \to \mathsf{set}$.
- The function symbol $\setminus$ (*set difference*), of arity $\mathsf{set} \times \mathsf{set} \to \mathsf{set}$.
- The predicate symbol $\in$ (*membership*), of arity $\mathsf{elem} \times \mathsf{set}$.

7.1.2 Definition
A standard set-structure is any Σ_{set}-structure $\mathcal{A}$ satisfying the following properties:

1. $A_{\mathsf{set}} = \mathcal{P}(A_{\mathsf{elem}})$.
2. the symbols $\varnothing$, $\{\cdot\}$, $\cup$, $\cap$, $\setminus$, $\in$, and $\subseteq$ are interpreted according to their standard interpretation over sets of elements.

7.1.3 Definition
The modelclass of sets is the pair $M_{\mathsf{set}} = (\Sigma_{\mathsf{set}}, \mathbf{A})$, where $\mathbf{A}$ is the class of all Σ_{set}-structures that are isomorphic to standard set-structures.

7.2 Equality reduction

7.2.1 Algorithm (IS-SATISFIABLE-SETS)
Input: A finite set Γ of flat Σ_{set}-literals
Output: `satisfiable` if Γ is satisfiable; `unsatisfiable` otherwise

function IS-SATISFIABLE-SETS(Γ)
  $\Delta \leftarrow$ SETS-TO-EQUALITY(Γ)
  return IS-SATISFIABLE-EQUALITY(Δ)
end function

function SETS-TO-EQUALITY(Γ)
  $X \leftarrow vars(\Gamma)$
  For each literal of the form $x \not\approx_{\mathsf{set}} y$ in Γ, generate a fresh elem-variable $w_{x,y}$. Let W be the set of freshly generated variables.
  Replace each formula of the form $x \not\approx_{\mathsf{set}} y$ in Γ with the formula

$$(w_{x,y} \in x \ \wedge \ w_{x,y} \notin y) \ \vee \ (w_{x,y} \notin x \ \wedge \ w_{x,y} \in y)$$

  Replace each formula of the form $x \approx \varnothing$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} e \notin x\,.$$

  Replace each formula of the form $x \approx \{e_0\}$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} [e \in x \ \leftrightarrow \ e = e_0]\,,$$

  Replace each formula of the form $x \approx y \cup z$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} [e \in x \ \leftrightarrow \ (e \in y \ \vee \ e \in z)]\,.$$

  Replace each formula of the form $x \approx y \cap z$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} [e \in x \ \leftrightarrow \ (e \in y \ \wedge \ e \in z)]\,.$$

  Replace each formula of the form $x \approx y \setminus z$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} [e \in x \ \leftrightarrow \ (e \in y \ \wedge \ e \notin z)]\,.$$

  return Γ
end function

7.2.2 Proposition
In algorithm IS-SATISFIABLE-SETS*, let Δ be the output returned by the call to* SETS-TO-EQUALITY*(Γ). Then the following are equivalent:*

1. *Γ is M_{set}-satisfiable.*

2. *Δ is $M^{\Omega}_{\approx}$-satisfiable, where $\Omega^{\mathrm{S}} = \{\mathsf{elem}, \mathsf{set}\}$, $\Omega^{\mathrm{C}} = \Omega^{\mathrm{F}} = \varnothing$, and $\Omega^{\mathrm{P}} = \{\in\}$.*

PROOF. ($1 \implies 2$). Immediate.

($1 \implies 2$). Let $\mathcal{B}$ be an Ω-interpretation over $X \cup W$ satisfying Δ. Then it is easy to check that Γ is satisfied by the M_{set}-interpretation $\mathcal{A}$ over X defined by letting

$$A_{\mathsf{elem}} = B_{\mathsf{elem}} \,,$$

and

$$e^{\mathcal{A}} = e^{\mathcal{B}} \,, \qquad \text{for all } e \in X_{\mathsf{elem}} \,,$$
$$x^{\mathcal{A}} = \left\{ e \in (X_{\mathsf{elem}} \cup W)^{\mathcal{A}} \mid \left(e, x^{\mathcal{B}}\right) \in \left(\in^{\mathcal{B}}\right) \right\} \,, \qquad \text{for all } x \in X_{\mathsf{set}} \,.$$

7.2.3 Proposition
Algorithm IS-SATISFIABLE-SETS *is correct.*

PROOF. Termination is obvious. Partial correctness follows by Proposition 7.2.2.