Chapter 8

Multisets

DECISION PROCEDURES
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8.1 The modelclass

8.1.1 Definition
The SIGNATURE OF MULTISETS Σ_{bag} extends the signature Σ_{int} of integers with the following symbols:

- The sorts elem for elements and bag for multisets of elements.
- The constant symbol $[\![\,]\!]$ (*empty bag*), of sort bag.
- The function symbol $[\![\cdot]\!]^{(\cdot)}$ (*singleton multiset*), of arity $\mathsf{elem} \times \mathsf{int} \to \mathsf{bag}$.
- The function symbol $\cup$ (*multiset union*), of arity $\mathsf{bag} \times \mathsf{bag} \to \mathsf{bag}$.
- The function symbol $\uplus$ (*multiset sum*), of arity $\mathsf{bag} \times \mathsf{bag} \to \mathsf{bag}$.
- The function symbol $\cap$ (*multiset intersection*), of arity $\mathsf{bag} \times \mathsf{bag} \to \mathsf{bag}$.
- The function symbol count (*counting*), of arity $\mathsf{bag} \times \mathsf{elem} \to \mathsf{int}$.

8.1.2 Definition
A STANDARD bag-STRUCTURE is any Σ_{bag}-structure $\mathcal{A}$ satisfying the following properties:

1. $\mathcal{A}^{\Sigma_{\mathsf{int}}}$ is the standard int-structure.
2. $A_{\mathsf{bag}} = \mathbb{N}^{A_{\mathsf{elem}}}$.
3. the symbols $[\![\,]\!]$, $[\![\cdot]\!]^{(\cdot)}$, $\cup$, $\uplus$, and $\cap$ are interpreted according to their standard interpretation over multisets of elements.

4. $\mathsf{count}^{\mathcal{A}}(e, x) = x(e)$, for all $e \in A_{\mathsf{elem}}$ and $x \in A_{\mathsf{bag}}$.

8.1.3 Definition

The MODELCLASS OF MULTISETS is the pair $M_{\mathsf{bag}} = (\Sigma_{\mathsf{bag}}, \mathbf{A})$, where $\mathbf{A}$ is the class of all Σ_{bag}-structures that are isomorphic to standard bag-structures.

8.2 Integer reduction

8.2.1 Algorithm (IS-SATISFIABLE-MULTISETS)

Input: A finite set Γ of flat Σ_{bag}-literals

Output: `satisfiable` if Γ is satisfiable; `unsatisfiable` otherwise

function IS-SATISFIABLE-SETS(Γ)
  $\Delta \leftarrow$ MULTISETS-TO-INTEGERS(Γ)
  return IS-SATISFIABLE-INTEGERS(Δ)
end function

function MULTISETS-TO-INTEGERS(Γ)
  $X \leftarrow \mathit{vars}(\Gamma)$
  For each literal of the form $x \not\approx_{\mathsf{bag}} y$ in Γ, generate a fresh elem-variable $w_{x,y}$. Let W be the set of freshly generated variables.
  Replace each literal of the form $x \not\approx_{\mathsf{bag}} y$ in Γ with the formula

$$\mathsf{count}(w_{x,y}, x) \not\approx \mathsf{count}(w_{x,y}, y)\,.$$

  Replace each formula of the form $x \approx [\![\,]\!]$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} \mathsf{count}(e, x) \approx 0\,.$$

  Replace each formula of the form $x \approx [\![e_0]\!]^{(u)}$ in Γ with the formula

$$\bigwedge_{e \in V_{\mathsf{elem}} \cup W} [\mathit{if}\ \ e \approx e_0\ \ \mathit{then}\ \mathsf{count}(e, x) \approx \mathsf{max}(0, u)\ \ \mathit{else}\ \mathsf{count}(e, x) \approx 0]\,.$$

  Replace each formula of the form $x \approx y \cup z$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} [\mathsf{count}(e, x) \approx \mathsf{max}(\mathsf{count}(e, y), \mathsf{count}(e, z))]\,.$$

  Replace each formula of the form $x \approx y \uplus z$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} [\mathsf{count}(e, x) \approx \mathsf{count}(e, y) + \mathsf{count}(e, z))]\,.$$

  Replace each formula of the form $x \approx y \cap z$ in Γ with the formula

$$\bigwedge_{e \in X_{\mathsf{elem}} \cup W} [\mathsf{count}(e, x) \approx \mathsf{min}(\mathsf{count}(e, y), \mathsf{count}(e, z))]\,.$$

  return Γ
end function

8.2.2 Proposition
In algorithm IS-SATISFIABLE-MULTISETS*, let* Δ *be the output returned by the call to* MULTISETS-TO-INTEGERS*(*Γ*). Then the following are equivalent:*

1. Γ *is* M_{bag}*-satisfiable.*
2. Δ *is* $M_{\mathsf{int}} \oplus M_{\approx}^{\Omega}$*-satisfiable, where* $\Omega^{\mathrm{S}} = \{\mathsf{int}, \mathsf{elem}, \mathsf{bag}\}$*,* $\Omega^{\mathrm{C}} = \varnothing$*,* $\Omega^{\mathrm{F}} = \{\mathsf{count}\}$*, and* $\Omega^{\mathrm{P}} = \varnothing$*.*

PROOF. ($1 \implies 2$). Immediate.

($1 \implies 2$). Let $\mathcal{B}$ be an $M_{\mathsf{int}} \oplus M_{\approx}^{\Omega}$-interpretation over $X \cup W$ satisfying Δ. Then it is easy to check that Γ is satisfied by the M_{bag}-interpretation $\mathcal{A}$ over X defined by letting

$$A_{\mathsf{elem}} = B_{\mathsf{elem}} \,,$$

and

$$\begin{aligned} e^{\mathcal{A}} &= e^{\mathcal{B}} \,, && \text{for all } e \in X_{\mathsf{elem}} \,, \\ u^{\mathcal{A}} &= u^{\mathcal{B}} \,, && \text{for all } u \in X_{\mathsf{int}} \,. \end{aligned}$$

Moreover, for each $a \in X_{\mathsf{bag}}$ and $e \in A_{\mathsf{elem}}$, we let

$$a^{\mathcal{A}}(e) = \begin{cases} \mathsf{count}^{\mathcal{B}}(e, a^{\mathcal{B}}) \,, & \text{if } e \in (X_{\mathsf{elem}} \cup W)^{\mathcal{B}} \,, \\ 0 \,, \text{otherwise} \,. \end{cases}$$

8.2.3 Proposition
Algorithm IS-SATISFIABLE-MULTISETS *is correct.*

PROOF. Termination is obvious. Partial correctness follows by Proposition 8.2.2.