Chapter 9

Combination

Decision Procedures
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9.1 Combination Theorem

9.1.1 Definition

Let $M_i = (\Sigma_i, \mathbf{A}_i)$ be a modelclass, for $i = 1, 2$. The combination of M_1 and M_2 is the modelclass $M_1 \oplus M_2 = (\Sigma, \mathbf{A})$ where $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\mathbf{A} = \{\mathcal{A} \mid \mathcal{A}^{\Sigma_1} \in \mathbf{A}_1 \text{ and } \mathcal{A}^{\Sigma_2} \in \mathbf{A}_2\}$.

9.1.2 Proposition

Let M be a Σ-modelclass, let $\mathcal{A}$ and $\mathcal{B}$ be Σ-interpretations over X, and let φ be a Σ-formula such that $vars(\varphi) \subseteq X$. Assume that $\mathcal{A} \cong \mathcal{B}$. Then

$$\mathcal{A} \models_M \varphi \qquad \Longleftrightarrow \qquad \mathcal{B} \models_M \varphi .$$

Proof. Immediate.

9.1.3 Proposition

For $i = 1, 2$, let M_i be a Σ_i-modelclass, let φ_i be a Σ_i-formula, and let $X_i = vars(\varphi_i)$. Also, let $\Sigma_0 = \Sigma_1 \cap \Sigma_2$ and $X_0 = X_1 \cap X_2$. Assume that there exist a Σ_1-interpretation $\mathcal{A}$ over X_1, and a Σ_2-interpretation $\mathcal{B}$ over X_2 such that:

$$\mathcal{A} \models_{M_1} \varphi_1 ,$$
$$\mathcal{B} \models_{M_2} \varphi_2 ,$$
$$\mathcal{A}^{\Sigma_0, X_0} \cong \mathcal{B}^{\Sigma_0, X_0} .$$

Then there exists a $(\Sigma_1 \cup \Sigma_2)$*-interpretation* $\mathcal{F}$ *over* $X_1 \cup X_2$ *such that:*

$$\begin{aligned} \mathcal{F} &\models_{M_1 \oplus M_2} \varphi_1 \wedge \varphi_2 \,, \\ \mathcal{F}^{\Sigma_1, X_1} &\cong \mathcal{A} \,, \\ \mathcal{F}^{\Sigma_2, X_2} &\cong \mathcal{B} \,. \end{aligned}$$

PROOF. Let h be an isomorphism of $\mathcal{A}^{\Sigma_0, X_0}$ into $\mathcal{B}^{\Sigma_0, X_0}$. By Proposition 9.1.2, we can assume without loss of generality that $\mathcal{A}^{\Sigma_0, X_0} = \mathcal{B}^{\Sigma_0, X_0}$. In particular, this implies that $A_\sigma = B_\sigma$, for all $\sigma \in \Sigma_0$.

We define a $(\Sigma_1 \cup \Sigma_2)$-interpretation $\mathcal{F}$ over $X_1 \cup X_2$ by letting:

$$F_\sigma = \begin{cases} A_\sigma \,, & \text{if } \sigma \in \Sigma_1^{\mathrm{S}} \,, \\ B_\sigma \,, & \text{if } \sigma \in \Sigma_2^{\mathrm{S}} \setminus \Sigma_1^{\mathrm{S}} \,, \end{cases}$$

and:

- for variables:

$$u^{\mathcal{F}} = \begin{cases} u^{\mathcal{A}} \,, & \text{if } u \in X_1 \,, \\ u^{\mathcal{B}} \,, & \text{if } u \in X_2 \setminus X_1 \,, \end{cases}$$

- for constant symbols:

$$c^{\mathcal{F}} = \begin{cases} c^{\mathcal{A}} \,, & \text{if } c \in \Sigma_1^{\mathrm{C}} \,, \\ c^{\mathcal{B}} \,, & \text{if } c \in \Sigma_2^{\mathrm{C}} \setminus \Sigma_1^{\mathrm{C}} \,, \end{cases}$$

- for function symbols:

$$f^{\mathcal{F}} = \begin{cases} f^{\mathcal{A}} \,, & \text{if } f \in \Sigma_1^{\mathrm{F}} \,, \\ f^{\mathcal{B}} \,, & \text{if } f \in \Sigma_2^{\mathrm{F}} \setminus \Sigma_1^{\mathrm{F}} \,, \end{cases}$$

- for predicate symbols:

$$p^{\mathcal{F}} = \begin{cases} p^{\mathcal{A}} \,, & \text{if } p \in \Sigma_1^{\mathrm{P}} \,, \\ p^{\mathcal{B}} \,, & \text{if } p \in \Sigma_2^{\mathrm{P}} \setminus \Sigma_1^{\mathrm{P}} \,. \end{cases}$$

By construction, $\mathcal{F}^{\Sigma_1, X_1} \cong \mathcal{A}$ and $\mathcal{F}^{\Sigma_2, X_2} \cong \mathcal{B}$. Thus, by Proposition 9.1.2, $\mathcal{F} \models_{M_1 \oplus M_2} \varphi_1 \wedge \varphi_2$.

9.1.4 Proposition

For $i = 1, 2$*, let* M_i *be a* Σ_i*-modelclass, let* φ_i *be a* Σ_i*-formula, and let* $X_i = vars(\varphi_i)$*. Assume that* $\Sigma_1^{\mathrm{C}} \cap \Sigma_2^{\mathrm{C}} = \Sigma_1^{\mathrm{F}} \cap \Sigma_2^{\mathrm{F}} = \Sigma_1^{\mathrm{P}} \cap \Sigma_2^{\mathrm{P}} = \varnothing$*. Finally, assume that there exist a* Σ_1*-interpretation* $\mathcal{A}$ *over* X_1 *and a* Σ_2*-interpretation* $\mathcal{B}$ *over* X_2 *such that:*

- $\mathcal{A} \models_{M_1} \varphi_1$*;*

- $\mathcal{B} \models_{M_2} \varphi_2$;
- $|A_\sigma| = |B_\sigma|$, *for all sorts* $\sigma \in \Sigma_1^{\mathrm{S}} \cap \Sigma_2^{\mathrm{S}}$;
- $x^{\mathcal{A}} = y^{\mathcal{A}}$ *iff* $x^{\mathcal{B}} = y^{\mathcal{B}}$, *for all variables* $x, y \in X_1 \cap X_2$ *of the same sort.*

Then there exists a $(\Sigma_1 \cup \Sigma_2)$*-interpretation* $\mathcal{F}$ *over* $X_1 \cup X_2$ *such that* $\mathcal{F} \models_{M_1 \oplus M_2} \varphi_1 \wedge \varphi_2$.

Proof. Let $X = X_1 \cap X_2$. For each sort $\sigma \in \Sigma_1^{\mathrm{S}} \cap \Sigma_2^{\mathrm{S}}$, define a function $h_\sigma : X_\sigma^{\mathcal{A}} \to X_\sigma^{\mathcal{B}}$ by letting $h_\sigma(x^{\mathcal{A}}) = x^{\mathcal{B}}$, for all variables $x \in X_\sigma$. By construction, h_σ is bijective. It follows that $|X_\sigma^{\mathcal{A}}| = |X_\sigma^{\mathcal{B}}|$. We can therefore extend h_σ to a bijective function $h'_\sigma : A_\sigma \to B_\sigma$. Thus, we have found a family of bijective functions

$$h = \left\{ h_\sigma : A_\sigma \to B_\sigma \mid \sigma \in \Sigma_1^{\mathrm{S}} \cap \Sigma_2^{\mathrm{S}} \right\} .$$

Clearly, h is an isomorphism of $\mathcal{A}^{\Sigma_1 \cap \Sigma_2, X}$ into $\mathcal{B}^{\Sigma_1 \cap \Sigma_2, X}$. By Proposition 9.1.3, there exists a $(\Sigma_1 \cup \Sigma_2)$-interpretation $\mathcal{F}$ over $X_1 \cup X_2$ such that $\mathcal{F} \models_{M_1 \oplus M_2} \varphi_1 \wedge \varphi_2$.

9.2 Nelson-Oppen

9.2.1 Definition
A Σ-modelclass M is stably infinite provided that every quantifier-free Σ-formula φ is M-satisfiable if and only if there exists a Σ-interpretation $\mathcal{A}$ over $vars(\varphi)$ such that $\mathcal{A} \models_M \varphi$ and A_σ is countably infinite, for every $\sigma \in \Sigma^{\mathrm{S}}$.

9.2.2 Definition
Let X be a set of variables, and let E be a well-sorted equivalence relation of X. The arrangement of X with respect to E is the set of literals

$$\begin{aligned} arr(E, X) = &\{x \approx y \mid (x, y) \in E\} \\ &\{x \not\approx y \mid (x, y) \notin E\} \end{aligned}$$

9.2.3 Algorithm (nelson-oppen)
Input: For $i = 1, 2$, M_i is a stably infinite Σ_i-modelclass with a decidable quantifier-free satisfiability problem. The algorithm takes in input a finite set Γ of $(\Sigma_1 \cup \Sigma_2)$-literals
Output: `satisfiable`, if Γ is $(M_1 \oplus M_2)$-satisfiable; `unsatisfiable` otherwise

function nelson-oppen(Γ)
    By opportunely introducing fresh variables, obtain a finite set

$$\Gamma_1 \cup \Gamma_2$$

    of literals, where Γ_i contains only Σ_i-literals, and such that Γ and $\Gamma_1 \cup \Gamma_2$ are $(M_1 \oplus M_2)$-equisatisfiable

$X \leftarrow vars(\Gamma_1) \cap vars(\Gamma_2)$
for all well-sorted equivalence relations E of X **do**
  if $\Gamma_1 \cup arr(X, E)$ is M_1-unsatisfiable **then**
    return `unsatisfiable`
  else if $\Gamma_2 \cup arr(X, E)$ is M_2-unsatisfiable **then**
    return `unsatisfiable`
  else
    return `satisfiable`
  end if
end for
end function

9.2.4 Proposition
If Algorithm NELSON-OPPEN *returns* `unsatisfiable` *then* Γ *is* $(M_1 \oplus M_2)$*-unsatisfiable.*

PROOF. By contradiction, assume that Γ is $(M_1 \oplus M_2)$-satisfiable. Then $\Gamma_1 \cup \Gamma_2$ is $(M_1 \oplus M_2)$-satisfiable. Let $\mathcal{F}$ be a $(\Sigma_1 \cup \Sigma_2)$-intepretation such that $\mathcal{F} \models_{M_1 \oplus M_2} \Gamma_1 \cup \Gamma_2$. Moreover, let E be the equivalence relation of X defined by $(x, y) \in E$ iff the variables x, y have the same sort and $x^{\mathcal{A}} = y^{\mathcal{A}}$. By construction, $\mathcal{F} \models \Gamma_1 \cup arr(X, E)$ and $\mathcal{F} \models \Gamma_2 \cup arr(X, E)$. This contradicts the fact that the algorithm returned `unsatisfiable`.

9.2.5 Proposition
If Algorithm NELSON-OPPEN *returns* `satisfiable` *then* Γ *is* $(M_1 \oplus M_2)$*satisfiable.*

PROOF. We know that, for $i = 1, 2$, $\Gamma_i \cup arr(X, E)$ is M_i-satisfiable. Since M_1 and M_2 are stably infinite, there exist a Σ_1-interpretation $\mathcal{A}$ and a Σ_2-interpretation $\mathcal{B}$ such that

- $\mathcal{A} \models_{M_1} \Gamma_1 \cup arr(X, E)$;
- $\mathcal{B} \models_{M_2} \Gamma_2 \cup arr(X, E)$;
- A_σ is countably infinite, for all sorts $\sigma \in \Sigma_1^{\mathrm{S}}$;
- B_σ is countably infinite, for all sorts $\sigma \in \Sigma_2^{\mathrm{S}}$.

Moreover, we have $x^{\mathcal{A}} = y^{\mathcal{A}}$ iff $x^{\mathcal{B}} = y^{\mathcal{B}}$, for all variables $x, y \in X$ of the same sort. Thus, by Proposition 9.1.4, there exists a $(\Sigma_1 \cup \Sigma_2)$-interpretation $\mathcal{F}$ such that $\mathcal{F} \models_{M_1 \oplus M_2} \Gamma_1 \cup \Gamma_2 \cup arr(X, E)$. It follows that $\Gamma_1 \cup \Gamma_2$ is $(M_1 \cup M_2)$-satisfiable, and therefore also Γ is is $(M_1 \cup M_2)$-satisfiable.

9.2.6 Proposition
Algorithm NELSON-OPPEN *is correct.*

PROOF. Termination is obvious. Partial correctness follows by Propositions 9.2.4 and 9.2.5.