Solving Games Via Three-Valued Abstraction Refinement

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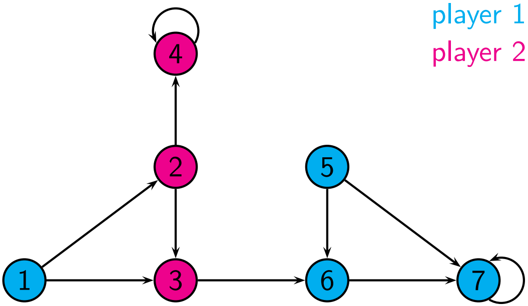
Introduction

- games are important for verification and synthesis
- problem: size of state-space
- solution: abstraction

Games

- *game structure* $G = (S, \lambda, \delta)$
- turn function $\lambda : S \to \{1, 2\}$ (so $S = S_1 \dot\cup S_2$)
- transition function $\delta : S \to 2^S \setminus \emptyset$

Example



$S = \{1, 2, 3, 4, 5, 6, 7\}$

Game Objectives

- *game objective* is an ω-regular language $\Phi \subseteq S^\omega$
- to win, sequence of states must be in this language
- here: reachability and safety
- reachability: $\Diamond T$ where $T \subseteq S$ denotes
  $\{\sigma \in S^\omega \mid \exists k \geq 0 . \sigma[k] \in T\}$
- safety: $\Box T$ where $T \subseteq S$ denotes
  $\{\sigma \in S^\omega \mid \forall k \geq 0 . \sigma[k] \in T\}$

Strategies

- *strategy* is a function $\pi_i : S^* \times S_i \to S$
- $\text{outcome}(s, \pi_1, \pi_2) = \sigma \in S^\omega$ such that
  $\forall k \geq 0.\ \sigma[k] \in S_i \implies \sigma[k+1] = \pi_i(\sigma[0..k])$
- s is *winning* for player 1 with objective Φ iff
  $\exists \pi_1 . \forall \pi_2 .\ \text{outcome}(s, \pi_1, \pi_2) \in \Phi$
- $\langle 1 \rangle \Phi := \{ s \in S \mid s \text{ is winning for player 1 with objective } \Phi \}$

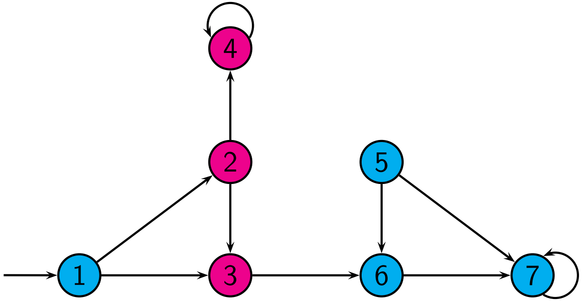
Controllable Predecessors

- $\mathsf{cpre}_1 : 2^S \to 2^S$
- $\mathsf{cpre}_1(T) = \{s \in S_1 \mid \delta(s) \cap T \neq \emptyset\} \cup \{s \in S_2 \mid \delta(s) \subseteq T\}$

Goal

- given game objective Φ
- given set of initial states $I \subseteq S$
- decide $I \cap \langle 1 \rangle \Phi \stackrel{?}{=} \emptyset$

Example



$\Phi = \Diamond\{7\}$

$\mathsf{cpre}_1(\{7\}) = \{5, 6, 7\}$

$\mathsf{cpre}_1(\{5, 6, 7\}) = \{3, 5, 6, 7\}$

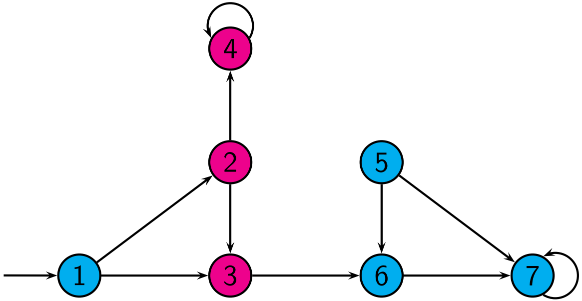
$\mathsf{cpre}_1(\{3, 5, 6, 7\}) = \{1, 3, 5, 6, 7\}$

$\langle 1 \rangle \Phi = \{1, 3, 5, 6, 7\}$

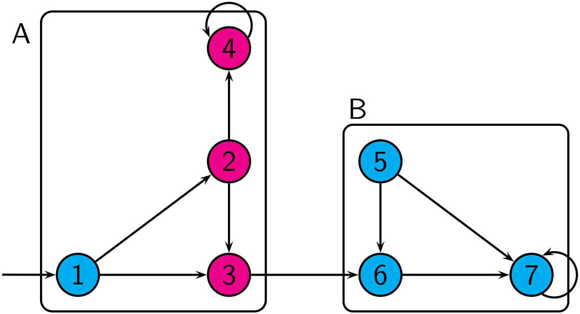
Abstractions

- an *abstraction* of $G = (S, \lambda, \delta)$ is a set $V \subseteq 2^S \setminus \{\emptyset\}$ of abstract states
- such that $\bigcup V = S$
- so each abstract state is a nonempty set of concrete states

Abstractions

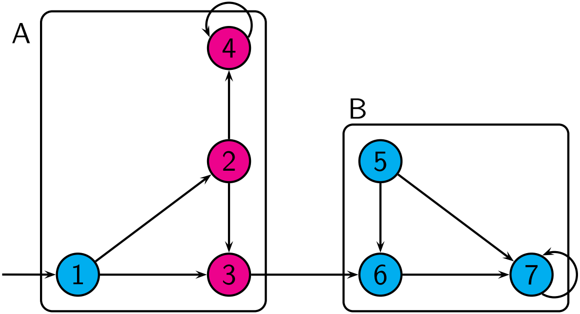


Abstractions



$$V = \{A, B\} = \{\{1, 2, 3, 4\}, \{5, 6\}\}$$

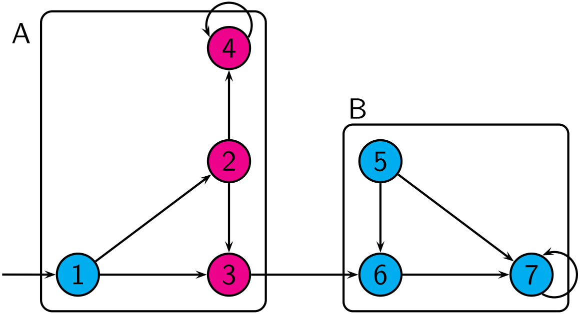
Abstractions



concrete states corresponding to a set U of abstract states:

$$U{\downarrow} := \bigcup_{u \in U} u$$

Abstractions

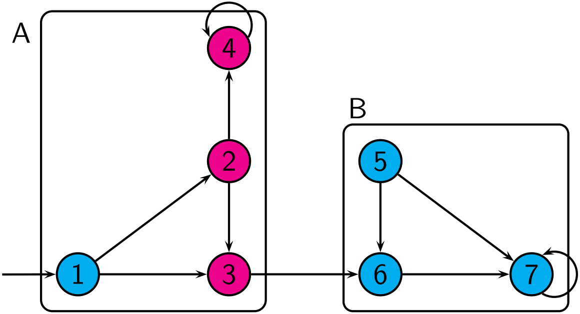


concrete states corresponding to a set U of abstract states:

$$U{\downarrow} := \bigcup_{u \in U} u$$

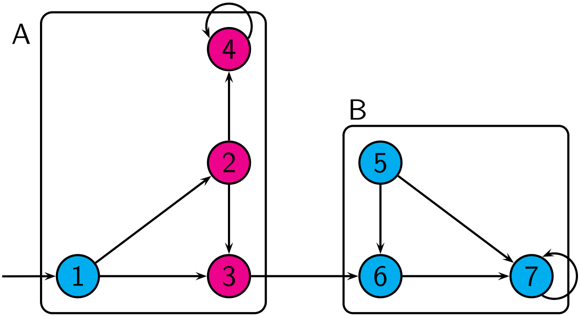
for instance: $\{B\}{\downarrow} = \{5, 6, 7\}$, $\{A, B\}{\downarrow} = S$

Abstractions



abstract states corresponding to a set T of concrete states?

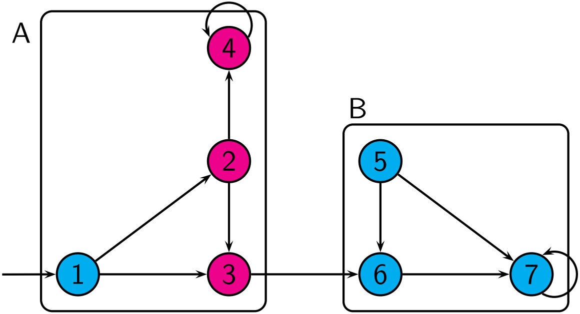
Abstractions



abstract states corresponding to a set T of concrete states?

- *under-approximation* $T^{\text{under}} := \{v \in V \mid v \subseteq T\}$
  e.g. $\{1\}^{\text{under}} = \emptyset$ and $\{1, 3, 5, 6, 7\}^{\text{under}} = \{B\}$

Abstractions



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  e.g. $\{1\}^{\text{under}} = \emptyset$ and $\{1, 3, 5, 6, 7\}^{\text{under}} = \{B\}$
- *over-approximation* $T^{\text{over}} := \{v \in V \mid v \cap T \neq \emptyset\}$
  e.g. $\{1\}^{\text{over}} = \{A\}$ and $\{1, 3, 5, 6, 7\}^{\text{over}} = \{A, B\}$

Abstractions

- for any $T \subseteq S$ we have $T^{\text{under}}\downarrow \subseteq T \subseteq T^{\text{over}}\downarrow$
- abstraction is *precise* for T iff $T^{\text{under}} = T^{\text{over}}$

Abstraction Refinement

- how to find a good abstraction?
- approach: *abstraction refinement*
- popular technique: CEGAR
- alternative proposal: three-valued analysis

Abstraction Refinement

- take abstraction
- compute must-win states, never-win states, and may-win states
- if not sufficiently precise: reduce number of may-win states and repeat
- refinement depends on the property in question!

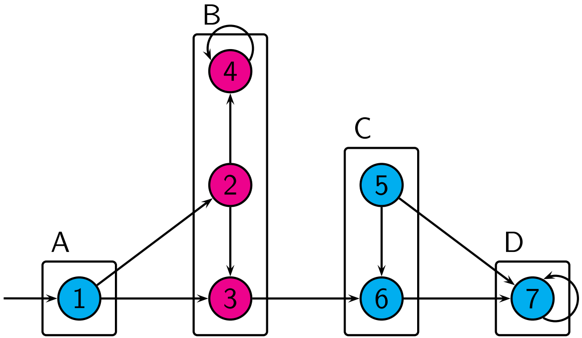
Abstraction Refinement

- in concrete game: state is winning if it's a cpre of a winning state
- in abstract game? approximate!

Algorithm for Reachability Games

```
while true do
    W_must := μY.(T^under ∪ cpre_1(Y↓)^under)
    W_may := μY.(T^over ∪ cpre_1(Y↓)^over)
    if W_may ∩ I^over = ∅ then return NO
    if W_must ∩ I^under ≠ ∅ then return YES
    choose v ∈ (W_may \ W_must) ∩ cpre_1(W_must↓)^over
    let v_1 = v ∩ cpre_1(W_must↓)
    let v_2 = v \ v_1
    V := (V \ {v}) ∪ {v_1, v_2}
done
```

Algorithm for Reachability Games



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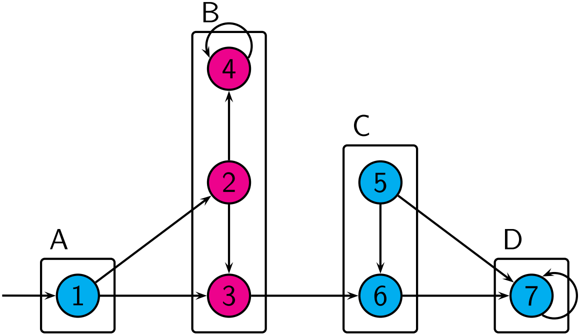
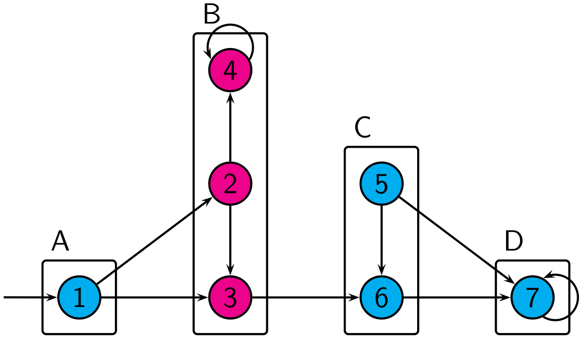


$$W_{\mathrm{must}} := \mu Y.(T^{\mathrm{under}} \cup \mathrm{cpre}_1(Y\!\downarrow)^{\mathrm{under}}) = \{C, D\}$$
$$W_{\mathrm{may}} := \mu Y.(T^{\mathrm{over}} \cup \mathrm{cpre}_1(Y\!\downarrow)^{\mathrm{over}}) = \{A, B, C, D\}$$

Algorithm for Reachability Games



if $W_{\text{may}} \cap I^{\text{over}} = \emptyset$ then return NO
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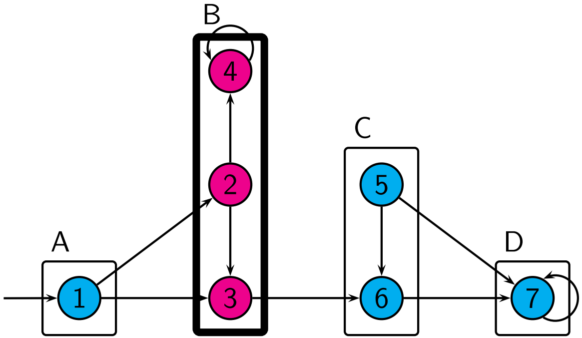
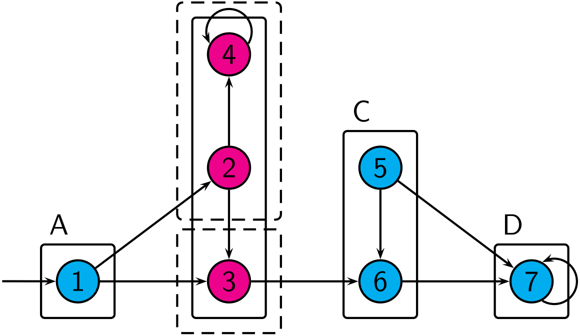
Algorithm for Reachability Games



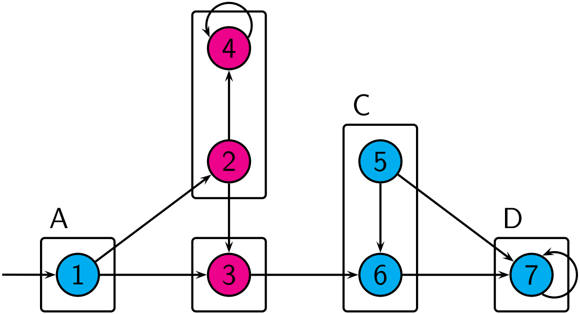
choose $v \in (W_{\mathsf{may}} \setminus W_{\mathsf{must}}) \cap \mathsf{cpre}_1(W_{\mathsf{must}}\!\downarrow)^{\mathsf{over}}$

Algorithm for Reachability Games



let $v_1 = v \cap \mathrm{cpre}_1(W_{\mathrm{must}}{\downarrow})$
let $v_2 = v \setminus v_1$

Algorithm for Reachability Games



$V := (V \setminus \{v\}) \cup \{v_1, v_2\}$

Algorithm for Safety Games

- dual to reachability: $\langle 1 \rangle \Box T = S \setminus \langle 2 \rangle \Diamond (S \setminus T)$

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Algorithm for Safety Games

- dual to reachability: $\langle 1 \rangle \Box T = S \setminus \langle 2 \rangle \Diamond (S \setminus T)$
- refinement for reachability:
  choose $v \in (W_{\text{may}} \setminus W_{\text{must}}) \cap \text{cpre}_1(W_{\text{must}}{\downarrow})^{\text{over}}$
- refinement for safety:
  choose $v \in (W_{\text{may}} \setminus W_{\text{must}}) \cap \text{cpre}_2(W^2_{\text{must}}{\downarrow})^{\text{over}}$
  i.e., $v \in (W_{\text{may}} \setminus W_{\text{must}}) \cap \text{cpre}_2(V \setminus W_{\text{may}}{\downarrow})^{\text{over}}$

Algorithm for Safety Games

while true do

$W_{\text{must}} := \nu Y.(T^{\text{under}} \cap \text{cpre}_1(Y{\downarrow})^{\text{under}})$

$W_{\text{may}} := \nu Y.(T^{\text{over}} \cap \text{cpre}_1(Y{\downarrow})^{\text{over}})$

if $W_{\text{may}} \cap I^{\text{over}} = \emptyset$ then return NO

if $W_{\text{must}} \cap I^{\text{under}} \neq \emptyset$ then return YES

choose $v \in (W_{\text{may}} \setminus W_{\text{must}}) \cap \text{cpre}_2(V \setminus W_{\text{may}}{\downarrow})^{\text{over}}$

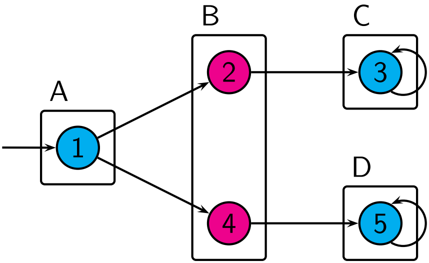
let $v_1 = v \cap \text{cpre}_2(S \setminus W_{\text{may}}{\downarrow})$

let $v_2 = v \setminus v_1$

$V := (V \setminus \{v\}) \cup \{v_1, v_2\}$

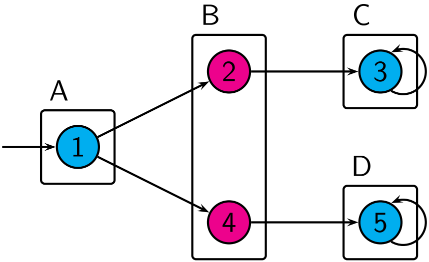
done

Algorithm for Safety Games



$\Phi = \Box\{1, 2, 3, 4\}$

Algorithm for Safety Games



$$W_{\text{must}} := \nu Y.(T^{\text{under}} \cap \text{cpre}_1(Y{\downarrow})^{\text{under}}) = \{\text{C}\}$$
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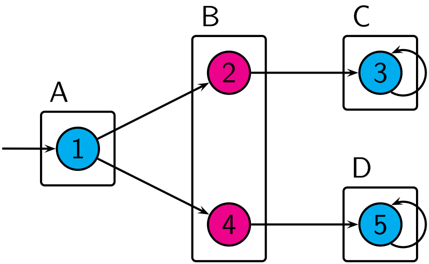
Algorithm for Safety Games



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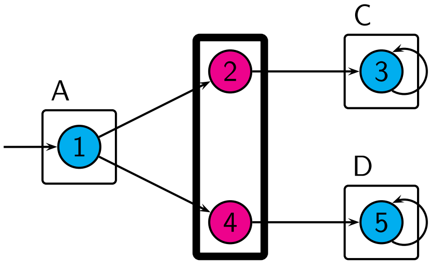
Algorithm for Safety Games



choose $v \in (W_{\text{may}} \setminus W_{\text{must}}) \cap \text{cpre}_2(V \setminus W_{\text{may}}{\downarrow})^{\text{over}}$

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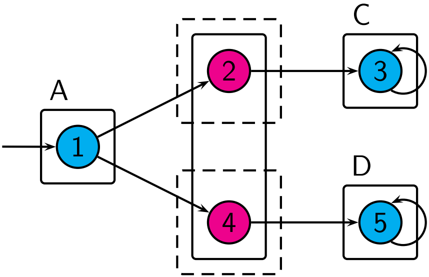
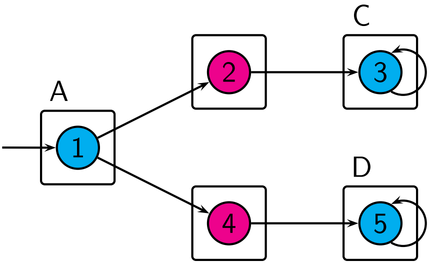


let $v_1 = v \cap \mathsf{cpre}_2(S \setminus W_{\mathsf{may}}\!\downarrow)$
let $v_2 = v \setminus v_1$

Algorithm for Safety Games



$$V := (V \setminus \{v\}) \cup \{v_1, v_2\}$$

Termination of the Algorithms

- correctness $\surd$
- termination?

Termination of the Algorithms

- correctness $\surd$
- termination? at least if there exists a finite region algebra for the game structure, i.e., an abstraction that is
  - closed under boolean operations
  - closed under controllable predecessor operators

Comparison to CEGAR

- 3-valued approach never needs more refinement steps
- however, CEGAR may need more than 3-valued approach
- reason is loss of precision due to abstract edges