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Infinite Games

Exercise Sheet 3

Deadline: 5 Nov. 2013

Exercise 3.1 - Attractor (1 + 1 + 2 + 2)

Let $\mathcal{A} = (V, V_0, V_1, E)$ be an arena and $R, R' \subseteq V$. Prove or disprove each of the following statements:

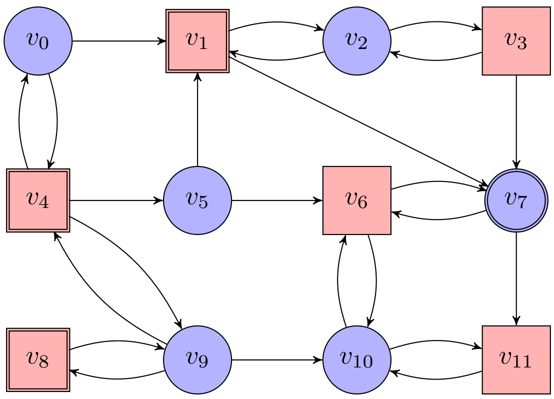
a) $\mathrm{CPre}_0(R) = V \setminus \mathrm{CPre}_1(V \setminus R)$

b) $R' \subseteq R \Rightarrow \mathrm{Attr}_0(R') \subseteq \mathrm{Attr}_0(R)$

c) $\mathrm{Attr}_0(R \cap R') = \mathrm{Attr}_0(R) \cap \mathrm{Attr}_0(R')$

d) $\mathrm{Attr}_0(R \cup R') = \mathrm{Attr}_0(R) \cup \mathrm{Attr}_0(R')$

Exercise 3.2 - Büchi (3 + 1)



Consider the Büchi game $\mathcal{G} = (\mathcal{A}, \text{BÜCHI}(F))$ depicted above. Compute the winning region and a corresponding uniform positional winning strategy for each Player i.

Exercise 3.3 - Traps (1 + 1 + 1 + 2)

A winning condition $\mathrm{Win} \subseteq V^\omega$ is prefix-independent, if we have for every $\rho \in V^\omega$ and every $w \in V^*$:

$$\rho \in \mathrm{Win} \text{ if and only if } w\rho \in \mathrm{Win}.$$

A set $T \subseteq V$ of vertices is a trap for Player i if all outgoing edges of the vertices in $V_i \cap T$ lead to T and at least one successor of every vertex in $V_{1-i} \cap T$ is in T.

a) Show: $V \setminus \mathrm{Attr}_i(R)$ is a trap for Player i for every set R

b) Prove or disprove: $\text{REACH}(R)$ is prefix-independent

c) Prove or disprove: $\text{BÜCHI}(F)$ is prefix-independent

d) Let $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$. Show: if Win is prefix-independent, then $W_0(\mathcal{G})$ and $W_1(\mathcal{G})$ are traps for Player 1 and Player 0, respectively

Exercise 3.4 - Challenge (2 Bonus Points)

In this exercise we want to consider games of incomplete information. Therefore, note the following game definition.

Definition E3.4.1. A reachability game $\mathcal{G}$ with incomplete information is a tuple (V, A, O, δ, o, R) consisting of

- a finite set V of vertices,
- a finite set A of actions,
- a finite set O of observations,
- a (non-deterministic) transition function $\delta : V \times A \to 2^V \setminus \{\emptyset\}$,
- an observation function $o : V \to O$ and
- a set $R \subseteq O$.

A play is an infinite alternating sequence $v_0\alpha_0v_1\alpha_1...$ of vertices and actions such that $v_{n+1} \in \delta(v_n, \alpha_n)$ for all $n \in \mathbb{N}$. The play is winning for Player 0, if there is an n such that $v_n \in R$.

A (pure) strategy for Player 0 is a mapping $\sigma : O^+ \to A$ while a (pure) strategy for Player 1 is a mapping $\tau : V^+ \times A \to V$ satisfying $\tau(wv, \alpha) \in \delta(v, \alpha)$ for every $wv \in V^+$ and every $\alpha \in A$. The play $v_0\alpha_0v_1\alpha_1...$ is consistent with σ, if $\alpha_n = \sigma(o(v_0)...o(v_n))$ for every n. It is consistent with τ, if $v_{n+1} = \tau(v_0...v_n, \alpha_n)$. So, intuitively, Player 0 picks an action based on the observations seen so far, while Player 1 resolves the non-determinism of the transition function. Note that Player 1 is fully informed. The strategy σ is a winning strategy for Player 0 from a vertex v, if every play that starts in v and is consistent with σ is winning for Player 0.

Show that the following problem is decidable:

> Given a reachability game $\mathcal{G} = (V, A, O, \delta, o, R)$ with incomplete information and a vertex $v \in V$. Does there exist a pure winning strategy for Player 0 from v?

(Note: Both players are aware that the play starts in v.)