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Exercise Sheet 6

Infinite Games

Deadline: 26 Nov. 2013

Exercise 6.1 - Reductions (3 + 5)

1. Recall the definition of challenging reachability games presented in Exercise 2.4. Show that such games are reducible to reachability games.

2. Let $\mathcal{A} = (V, V_0, V_1, E)$ be an arena. Given a finite family $(Q_j, P_j)_{j\in[k]}$ of subsets $Q_j, P_j \subseteq V$, we define the request-response condition $\textsc{reqres}((Q_j, P_j)_{j\in[k]})$ as

$$\textsc{reqres}((Q_j, P_j)_{j\in[k]}) = \{\, \rho \in \mathrm{Plays}(\mathcal{A}) \mid \forall j, n \in \mathbb{N}.\ \rho_n \in Q_j \Rightarrow \exists m \geq n.\ \rho_m \in P_j \,\}$$

   Intuitively, a visit to Q_j is a request that has to be answered by a later response, i.e., a visit to P_j. A game $\mathcal{G} = (\mathcal{A}, \textsc{reqres}((Q_j, P_j)_{j\in[k]}))$ is called a request-response game.

   Show that request-response games are reducible to Büchi games.

Exercise 6.2 - Reduction Lemma, revisited (4)

Let $\mathcal{G} \leq_{\mathcal{M}} \mathcal{G}'$ for two games $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ with $\mathcal{A} = (V, V_0, V_1, E)$ and $\mathcal{G}' = (\mathcal{A} \times \mathcal{M}, \mathrm{Win}')$, where $\mathcal{M} = (M, \mathrm{init}, \mathrm{upd})$. Furthermore, let $V' \subseteq V$. Show that if Player i has a finite-state winning strategy from $\{\, (v, \mathrm{init}(v)) \mid v \in V' \,\}$, implemented by some memory structure $\mathcal{M}'$, then she also has a finite-state winning strategy from V' for $\mathcal{G}$.

Hint: Start by defining a suitable product memory structure $\mathcal{M} \times \mathcal{M}'$.

Exercise 6.3 - Uniform Finite-state Strategies (3)

Prove or disprove: If Player i has a finite-state winning strategy from each vertex $v \in W_i(\mathcal{G})$, in an arbitrary game $\mathcal{G}$, then Player i has a uniform finite-state winning strategy for $\mathcal{G}$.

Exercise 6.4 - Challenge (2 Bonus Points)

Consider a request-response game $\mathcal{G} = (\mathcal{A}, \textsc{reqres}((Q_j, P_j)_{j\in[k]}))$ with $\mathcal{A} = (V, V_0, V_1, E)$ as defined in Exercise 6.1. We define the waiting time for condition j, denoted by $\mathrm{wt}_j \colon V^* \to \mathbb{N}$, recursively for $w, w' \in V^*$ and $v \in V$ as follows

$$\mathrm{wt}_j(w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 & \text{if } w = w'v \wedge \mathrm{wt}_j(w') = 0 \wedge v \in Q_j \setminus P_j \\ 0 & \text{if } w = w'v \wedge \mathrm{wt}_j(w') = 0 \wedge v \notin Q_j \setminus P_j \\ \mathrm{wt}_j(w') + 1 & \text{if } w = w'v \wedge \mathrm{wt}_j(w') > 0 \wedge v \notin P_j \\ 0 & \text{if } w = w'v \wedge \mathrm{wt}_j(w') > 0 \wedge v \in P_j \end{cases}$$

Intuitively, wt_j measures the waiting time between a request and its (earliest) response (ignoring additional requests of condition j while wt_j is already non-zero). The waiting times measure the quality of plays and strategies. Give a family of request-response games $\mathcal{G}_k = (\mathcal{A}_k, \textsc{reqres}((Q_j, P_j)_{j\in[k]}))$ with $|\mathcal{A}_k| \in \mathcal{O}(k)$ and k pairs (Q_j, P_j) such that every $\mathcal{A}_k$ has a distinguished vertex v satisfying:

- Player 0 has a winning strategy from v, but
- every winning strategy for Player 0 from v has a play $\rho \in \mathrm{Plays}(\mathcal{A}_k, \sigma, v)$ satisfying $\mathrm{wt}_j(\rho[n]) \geq 2^k$ for some $n \in \mathbb{N}$.

Can you derive an upper bound on wt_j from your reduction in Exercise 6.1? Does this bound (asymptotically) match the lower bound 2^k?