Dr. Martin Zimmermann
Felix Klein, B.Sc.

Infinite Games

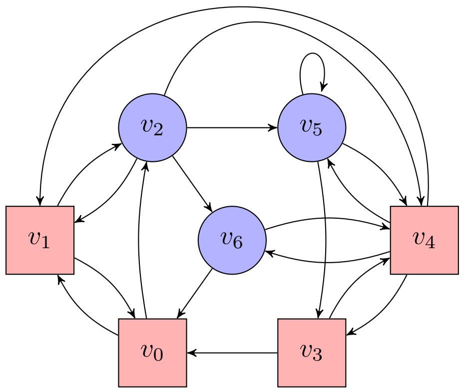
Exercise Sheet 7

Deadline: 3 Dec. 2013

Exercise 7.1 - Muller (3 + 2)

a) Consider the Muller game $\mathcal{G}_1 = (\mathcal{A}_1, \textsc{muller}(\mathcal{F}_1))$ with $\mathcal{A}_1$ as depicted below and

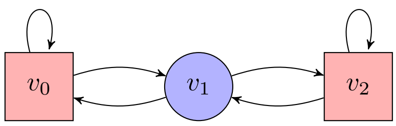
$$\mathcal{F}_1 = \{\{v_0, v_1\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_4, v_5, v_6\}, \{v_0, v_1, v_2, v_6\}\}.$$



Determine the winning regions of $\mathcal{G}_1$ and uniform finite-state winning strategies for both players. Specify the strategies by giving a memory structure (not necessarily the same for both players) and a next-move function.

b) Consider the Muller game $\mathcal{G}_2 = (\mathcal{A}_2, \textsc{muller}(\mathcal{F}_2))$ with $\mathcal{A}_2$ as depicted below and

$$\mathcal{F}_2 = \{\{v_0\}, \{v_2\}, \{v_0, v_1, v_2\}\}.$$



Apply the LAR reduction to determine the winning regions of $\mathcal{G}_2$, where constructing the vertices reachable from $\{\, (v, \mathrm{init}(v)) \mid v \in \{v_0, v_1, v_2\}\,\}$ suffices.

Exercise 7.2 - Union-closed Muller (5)

A family $\mathcal{F} \subseteq 2^V$ of sets is union-closed, if $F \cup F' \in \mathcal{F}$ for all $F, F' \in \mathcal{F}$. A Muller game is doubly union-closed, if $\mathcal{F}$ and $2^V \setminus \mathcal{F}$ are union-closed.

Show that doubly union-closed Muller games are equivalent to parity games, i.e.

- for every doubly union-closed Muller game $(\mathcal{A}, \textsc{muller}(\mathcal{F}))$ there exists a parity game $(\mathcal{A}, \textsc{parity}(\Omega))$ such that $\textsc{muller}(\mathcal{F}) = \textsc{parity}(\Omega)$ and
- for every parity game $(\mathcal{A}, \textsc{parity}(\Omega))$ there exists a doubly union-closed Muller game $(\mathcal{A}, \textsc{muller}(\mathcal{F}))$ such that $\textsc{parity}(\Omega) = \textsc{muller}(\mathcal{F})$.

Exercise 7.3 - ω-regular Games (1 + 2 + 2)

A (deterministic word) parity automaton $\mathscr{A} = (Q, \Sigma, q_I, \delta, \Omega)$ is a tuple consisting of

- a finite set Q of states,
- an alphabet Σ,
- an initial state $q_I \in Q$,
- a transition function $\delta\colon Q \times \Sigma \to Q$ and
- a coloring function $\Omega\colon Q \to [k]$.

The run $r = r_0 r_1 r_2 ... \in Q^\omega$ of $\mathscr{A}$ on an infinite input word $\alpha \in \Sigma^\omega$ is defined by $r_0 = q_I$ and $r_{n+1} = \delta(r_n, \alpha_n)$ for all $n \in \mathbb{N}^+$. The run is accepting iff $\mathrm{Par}(min(\Omega(\inf(r)))) = 0$. The language $\mathcal{L}(\mathscr{A})$ of $\mathscr{A}$ is the set of all input words whose run is accepting. A game $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ is ω-regular iff there exists a parity automaton $\mathscr{A}$ with $\mathcal{L}(\mathscr{A}) = \mathrm{Win}$. Prove the following statements:

a) Parity games are ω-regular.

b) Muller games are ω-regular.

c) Every ω-regular game is determined with uniform finite-state winning strategies.

Exercise 7.4 - Challenge (2 Bonus Points)

Show that every uniform finite-state winning strategy for Player 0 in the game DJW_n has at least $n!$ many memory states.

Hint: You can prove this by induction over $n \in \mathbb{N}$. Use the fact that DJW_{n+1} contains $n + 1$ different copies of DJW_n.