Dr. Martin Zimmermann
Felix Klein, B.Sc.

Exercise Sheet 8

Infinite Games

Deadline: 10 Dec. 2013

Exercise 8.1 - Closure Properties (4)

Show that $\mathbf{\Sigma}_n$ and $\mathbf{\Pi}_n$ are closed under union and intersection for every $n \in \mathbb{N}^+$.

Exercise 8.2 - Borel Hierarchy (2 + 2 + 2)

Let V be some finite set. Prove each membership in the Borel hierarchy stated below.

a) $\textsc{wmuller}(\mathcal{F}) = \{ \rho \in V^\omega \mid \mathrm{Occ}(\rho) \in \mathcal{F} \} \in \mathbf{\Sigma}_2 \cap \mathbf{\Pi}_2$ for every $\mathcal{F} \subseteq 2^V$.

b) $\textsc{cobüchi}(C) = \{ \rho \in V^\omega \mid \mathrm{Inf}(\rho) \subseteq C \} \in \mathbf{\Sigma}_2$ for every $C \subseteq V$.

c) $\textsc{parity}(\Omega) = \{ \rho \in V^\omega \mid \mathrm{Par}(\min(\Omega(\mathrm{Inf}(\rho)))) = 0 \} \in \mathbf{\Sigma}_3 \cap \mathbf{\Pi}_3$ for every $\Omega \colon V \to \mathbb{N}$.

Hint: Use the closure properties from Exercise 8.1.

Exercise 8.3 - Wadge Games (2 + 2 + 1)

A language $L \subseteq \mathbb{B}^\omega$ is *complete* for a level $\mathbf{\Sigma}_n$ of the Borel hierarchy over $\mathbb{B}$ iff $L \in \mathbf{\Sigma}_n$ and $L' \leq L$ for every $L' \subseteq \mathbb{B}^\omega$ with $L' \in \mathbf{\Sigma}_n$. Completeness for $\mathbf{\Pi}_n$ is defined similarly.

a) Show that $0^*1(0+1)^\omega$ is complete for $\mathbf{\Sigma}_1$.

b) Show that $(0^*1)^\omega$ is complete for $\mathbf{\Pi}_2$.

c) Show that $(0^*1)^\omega$ is not in $\mathbf{\Sigma}_1 \cup \mathbf{\Pi}_1$.

Exercise 8.4 - Challenge (2 Bonus Points)

Let $\mathcal{G} = (\mathcal{A}, \textsc{parity}(\Omega \colon V \to [k]))$ be a parity game. Show how you can construct a safety game $\mathcal{G}_s = (\mathcal{A} \times \mathcal{M}, \textsc{safe}(S))$ for some memory structure $\mathcal{M} = (M, \mathrm{init}, \mathrm{upd})$ such that for all $v \in V$ it holds that $v \in W_0(\mathcal{G}) \Leftrightarrow (v, \mathrm{init}(v)) \in W_0(\mathcal{G}_s)$.

Hint: Revisit the small progress measure algorithm for parity games.