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Infinite Games

Exercise Sheet 9

Deadline: 17 Dec. 2013

Exercise 9.1 - Colorful Max-Parity (4)

Let $\mathcal{A} = (V, V_0, V_1, E)$ be a, possibly countably infinite, arena. Given a coloring function $\Omega\colon V \to \mathbb{N}$ and $i_\emptyset, i_\infty \in \{0,1\}$ we define the max-parity condition as

$$\textsc{maxParity}(\Omega, i_\emptyset, i_\infty) = \{\, \rho \in \mathrm{Plays}(\mathcal{A}) \mid \mathrm{Inf}(\Omega(\rho)) \text{ is finite} \wedge \mathrm{Par}(max(\mathrm{Inf}(\Omega(\rho)))) = 0 \,\} \cup P_\emptyset \cup P_\infty$$

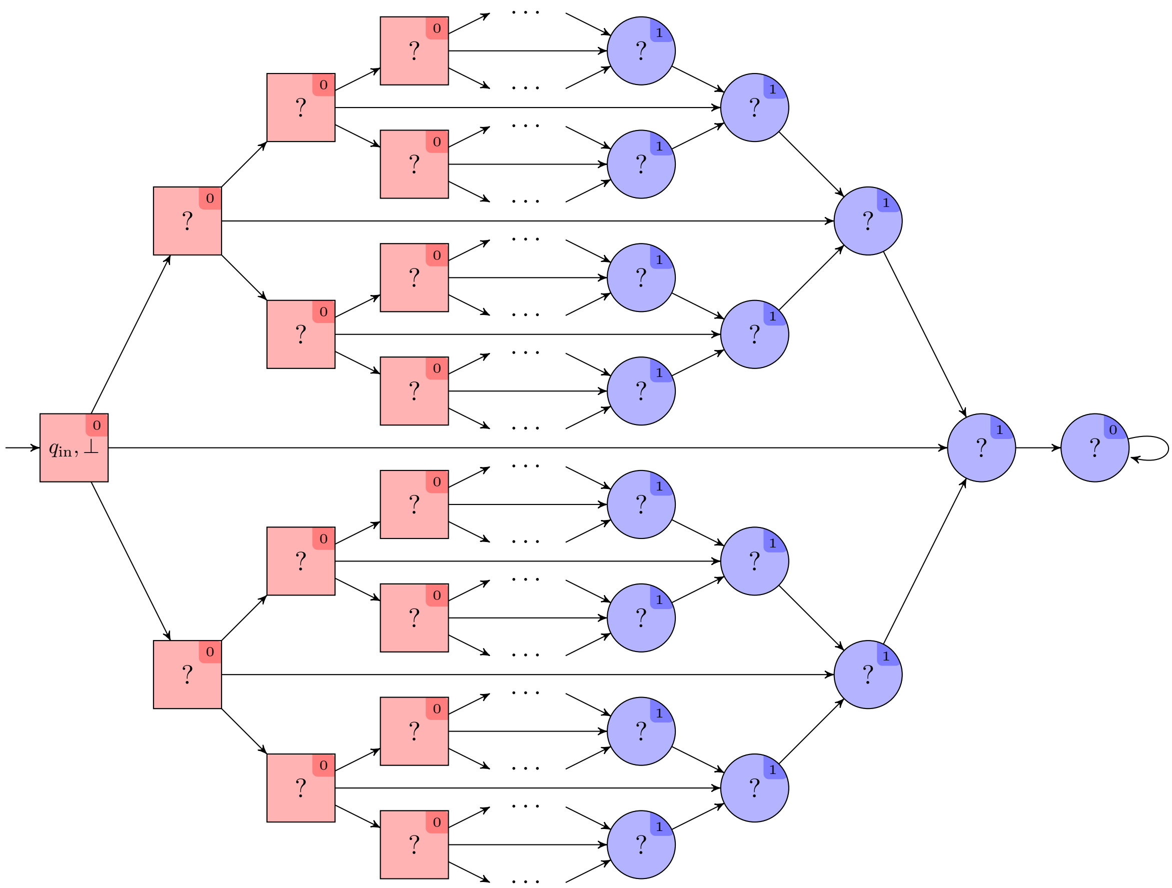
where $P_\emptyset$ and P_∞ are defined as

$$P_\emptyset = \begin{cases} \emptyset & \text{if } i_\emptyset = 1 \\ \{\, \rho \mid \mathrm{Inf}(\Omega(\rho)) = \emptyset \,\} & \text{if } i_\emptyset = 0 \end{cases} \quad \text{and} \quad P_\infty = \begin{cases} \emptyset & \text{if } i_\infty = 1 \\ \{\, \rho \mid \mathrm{Inf}(\Omega(\rho)) \text{ is infinite} \,\} & \text{if } i_\infty = 0 \end{cases}$$

Here, $\Omega(\rho)$ denotes the sequence of colors seen during ρ, i.e. $\Omega(\rho) = \Omega(\rho_0)\Omega(\rho_1)\Omega(\rho_2)\ldots \in \Omega(V)^\omega$. Accordingly, if only finitely many colors are seen infinitely often, then the parity of the maximal one determines the winner. If no color is seen infinitely often, then Player $i_\emptyset$ wins, while Player i_∞ wins if infinitely many colors are seen infinitely often.

Give an arena $\mathcal{A} = (V, V_0, V_1, E)$ and a coloring function $\Omega\colon V \to \mathbb{N}$ such that the winning regions of the games $\mathcal{G}_{i_\emptyset, i_\infty} = (\mathcal{A}, \textsc{maxParity}(\Omega, i_\emptyset, i_\infty))$ are pairwise different for every value of $(i_\emptyset, i_\infty)$, that is if $(i_\emptyset, i_\infty) \neq (i'_\emptyset, i'_\infty)$ then $W_0(\mathcal{G}_{i_\emptyset, i_\infty}) \neq W_0(\mathcal{G}_{i'_\emptyset, i'_\infty})$.

Exercise 9.2 - Pushdown Systems and Transducers (2 + 4)



a) Give a pushdown system, a partition of its states and a coloring function inducing the parity game sketched above, where only the part reachable from $(q_{in}, \bot)$ is sketched.

b) Give a pushdown transducer implementing a winning strategy for one of the players from $(q_{in}, \bot)$.

Exercise 9.3 - Pushdown Parity Games **(2 + 3)**

Consider the family of pushdown systems $\mathcal{P}_n = (Q_n, \{A\}, \Delta_n, q_{\text{in}})$ for $n \in \mathbb{N}^+$ with Q_n and Δ_n defined as follows.

- $Q_n = \{q_{\text{in}}, q_A\} \cup \bigcup_{j=1}^{n} \{\, q_j^t \mid t \in [p_j] \,\}$

  where p_j denotes the j-th prime number, i.e. $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ...

- $\Delta_n = \{\, (q_{\text{in}}, X, q_{\text{in}}, AX), (q_{\text{in}}, X, q_A.AX) \mid X \in \{A, \bot\} \,\}$
  $\cup \{\, (q_A, A, q_j^0, A) \mid j \in [1, n] \,\}$
  $\cup \{\, (q_j^t, A, q_j^{t'}, \varepsilon) \mid j \in [1, n],\ t \in [p_j],\ t' = (t+1) \bmod p_j \,\}$
  $\cup \{\, (q, \bot, q, \bot) \mid q \in Q_n \setminus \{q_{\text{in}}\} \,\}$

Further, let $Q_0 = Q_n \setminus \{q_A\}$ and $Q_1 = \{q_A\}$ be a partition of Q_n with $n \in \mathbb{N}^+$ and let $\Omega\colon Q_n \to [2]$ be the coloring function defined by

$$\Omega(q) = \begin{cases} 0 & \text{if } \exists j \in [1, n].\ q = q_j^0 \\ 1 & \text{otherwise} \end{cases}$$

Let $\mathcal{G}_n$ be the pushdown parity game induced by $\mathcal{P}_n$ and the partition and coloring defined above.

a) Draw $\mathcal{G}_2$ up to stack height 8 and give a positional winning strategy from $(q_{\text{in}}, \bot)$ for one of the players.

b) Give a winning strategy from $(q_{\text{in}}, \bot)$ for one of the players for every $\mathcal{P}_n$ with $n \in \mathbb{N}^+$.

Exercise 9.4 - Challenge **(2 Bonus Points)**

Show how to solve pushdown parity games that are induced by a pushdown system without pop transitions, some partition and some coloring function.