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Infinite Games

Exercise Sheet 10

Deadline: 7 Jan. 2014

Exercise 10.1 - Recap (15)

1. Solving Muller games is in NP $\cap$ Co-NP if $\mathcal{F}$ is encoded

☐ by a circuit? ☐ by a coloring function? ☐ by a tree? ☐ by an important subset? ☐ by a boolean formula?

2. In which of the following games is W_0 a trap for Player 1?

☐ Reachability ☐ Request Response ☐ Parity ☐ Safety ☐ Büchi ☐ Challenging Reachability ☐ Weak Muller

3. In a game with n vertices, which of the following winning conditions has the largest memory requirements for Player 0? (for sufficiently large n)

☐ Weak Muller ☐ Weak Parity ☐ Büchi ☐ Muller ☐ Parity ☐ Challenging Reachability ☐ Safety

4. What is the lower bound on the size of a winning strat. for Pl. 0 in weak Muller games?

☐ $\mathcal{O}(\log|\mathcal{F}|)$ ☐ $\mathcal{O}(|V|\cdot|\mathcal{F}|)$ ☐ $\mathcal{O}(2^{2^{|V|}})$ ☐ $2^{\mathcal{O}(|V|)}$ ☐ $\mathcal{O}(|V|^2)$ ☐ $\mathcal{O}(\log^*(|V|))$ ☐ 4

5. Which of the following winning conditions can be described by a parity condition?

☐ SAFE ☐ REQRES ☐ WMULLER ☐ BÜCHI ☐ MULLER($\mathcal{F}$) with $\mathcal{F}$ doubly union-closed ☐ REACH ☐ CHREACH ☐ MULLER

6. Which of the following games are known to be solvable in polynomial time?

☐ Pushdown ☐ Bipartite Parity ☐ Büchi ☐ Solitary Parity ☐ Weak Parity ☐ Challenging Reachability ☐ Parity

7. Which of the following statements hold?

☐ BÜCHI(F) $\in \mathbf{\Sigma}_2$ ☐ PARITY(Ω) $\in \mathbf{\Sigma}_2$ ☐ WMULLER($\mathcal{F}$) $\in \mathbf{\Sigma}_2 \cap \mathbf{\Pi}_2$ ☐ SAFE(S) $\in \mathbf{\Sigma}_2 \cap \mathbf{\Pi}_2$ ☐ REQRES($(Q_j, P_j)_{j\in[k]}$) $\in \mathbf{\Sigma}_3$ ☐ BÜCHI(F) $\in \mathbf{\Pi}_2$ ☐ MULLER($\mathcal{F}$) $\in \mathbf{\Pi}_2$ ☐ REACH(R) $\in \mathbf{\Pi}_1$

8. Which of the following winning conditions are prefix independent?

☐ WMULLER ☐ COBÜCHI ☐ BÜCHI ☐ REQRES ☐ MULLER ☐ PARITY ☐ SAFE

9. For which of the following games has Player 1 positional winning strategies?

☐ Muller ☐ Büchi ☐ Reachability ☐ co-Büchi ☐ Weak Parity ☐ Safety ☐ Parity ☐ Challenging Reachability

10. In which of the following games can Player 0 win with a uniform strategy?

☐ Muller ☐ Reachability ☐ Challenging Reachability ☐ co-Büchi ☐ Weak Parity ☐ Request-Response ☐ Parity

11. Let $\mathcal{G} = (\mathcal{A}, \text{PARITY}(\Omega\colon V \to [k])$ be a parity game with $\text{Par}(k) = 0$. How large is $|\text{Sh}(\mathcal{G})|$?

☐ $|V|$ ☐ $2^{|V|}$ ☐ $|V|^2$ ☐ $1 + \prod_{c\in\Omega(V),\, \text{Par}(c)=1} |\Omega^{-1}(c)|$ ☐ $\frac{|V|}{2}! + 1$ ☐ 42 ☐ $|V|\cdot|E|$

12. Which of the following games are self-dual?

☐ co-Büchi ☐ Request-Response ☐ Reachability ☐ Parity ☐ Challenging Reachability ☐ Weak Muller ☐ Weak Parity

13. In which of the following games may Player 0 need memory?

☐ Büchi ☐ Weak Parity ☐ Safety ☐ Weak Muller ☐ Parity ☐ Request-Response ☐ Muller

14. Using game reductions, can you reduce

☐ co-Büchi to Reachability? ☐ Büchi to Safety? ☐ Safety to Reachability? ☐ Parity to Muller? ☐ Muller to Büchi? ☐ Request-Response to Parity? ☐ Büchi to Parity?

15. Which of the following games are determined?

☐ Muller ☐ Safety ☐ Request-Response ☐ Parity ☐ Büchi ☐ Reachability ☐ co-Büchi

Exercise 10.2 - Zielonka Trees, Rabin, Streett (10 Bonus Points)

Given a family $\mathcal{F} \subseteq 2^V$ of subsets of a finite set V, we define its Zielonka tree $\mathcal{Z}(\mathcal{F})$ recursively as follows:

- The root of $\mathcal{Z}(\mathcal{F})$ is labeled by the set of all vertices.
- Children of a node labeled with $F \in \mathcal{F}$ are the $\subseteq$-maximal subsets $F' \subseteq F$ with $F' \notin \mathcal{F}$.
- Children of a node labeled with $F \notin \mathcal{F}$ are the $\subseteq$-maximal subsets $F' \subseteq F$ with $F' \in \mathcal{F}$.

We already had an example of such a tree on page 39 of the lecture notes. We say that a vertex v of $\mathcal{Z}(\mathcal{F})$ is a Player 0 vertex if and only if its label is in $\mathcal{F}$.

Given a family $(Q_j, P_j)_{j\in[k]}$ of subsets $Q_j, P_j \subseteq V$ with $k \in \mathbb{N}$ we define the Rabin winning condition by

$$\textsc{rabin}((Q_j, P_j)_{j\in[k]}) = \{\, \rho \in V^\omega \mid \exists j \in [k].\ \mathrm{Inf}(\rho) \cap Q_j \neq \emptyset \wedge \mathrm{Inf}(\rho) \cap P_j = \emptyset \,\}$$

and the Streett winning condition by

$$\textsc{streett}((Q_j, P_j)_{j\in[k]}) = \{\, \rho \in V^\omega \mid \forall j \in [k].\ \mathrm{Inf}(\rho) \cap Q_j \neq \emptyset \Rightarrow \mathrm{Inf}(\rho) \cap P_j \neq \emptyset \,\}$$

Given an arena $\mathcal{A} = (V, V_0, V_1, E)$ we then call the games $\mathcal{G}_r = (\mathcal{A}, \textsc{rabin}((Q_j, P_j)_{j\in[k]}))$ and $\mathcal{G}_s = (\mathcal{A}, \textsc{streett}((Q_j, P_j)_{j\in[k]}))$ a Rabin game and a Street game, respectively.

Prove the following statements:

a) For every family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ holds that

$$\textsc{rabin}((Q_j, P_j)_{j\in[k]}) = V^\omega \setminus \textsc{streett}((Q_j, P_j)_{j\in[k]}).$$

b) For every coloring function $\Omega\colon V \to \mathbb{N}$ there exists a family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ such that $\textsc{parity}(\Omega) = \textsc{rabin}((Q_j, P_j)_{j\in[k]})$.

c) For every coloring function $\Omega\colon V \to \mathbb{N}$ there exists a family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ such that $\textsc{parity}(\Omega) = \textsc{streett}((Q_j, P_j)_{j\in[k]})$.

d) For every family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ there is a set $\mathcal{F} \subseteq 2^V$ such that $\textsc{rabin}((Q_j, P_j)_{j\in[k]}) = \textsc{muller}(\mathcal{F})$.

e) For every family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ there is a set $\mathcal{F} \subseteq 2^V$ such that $\textsc{streett}((Q_j, P_j)_{j\in[k]}) = \textsc{muller}(\mathcal{F})$.

f) For every set $\mathcal{F} \subseteq 2^V$ holds: every Player 0 vertex of $\mathcal{Z}(\mathcal{F})$ has at most one successor if and only if $\textsc{muller}(\mathcal{F}) = \textsc{rabin}((Q_j, P_j)_{j\in[k]})$ for some family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$.

g) For every set $\mathcal{F} \subseteq 2^V$ holds: every Player 1 vertex of $\mathcal{Z}(\mathcal{F})$ has at most one successor if and only if $\textsc{muller}(\mathcal{F}) = \textsc{streett}((Q_j, P_j)_{j\in[k]})$ for some family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$.

h) For every set $\mathcal{F} \subseteq 2^V$ holds: every vertex of $\mathcal{Z}(\mathcal{F})$ has at most one successor if and only if $\textsc{muller}(\mathcal{F}) = \textsc{parity}(\Omega)$ for some coloring function $\Omega\colon V \to \mathbb{N}$.

i) Let $\mathcal{Z}(\mathcal{F})$ be the Zielonka tree for some $\mathcal{F} \subseteq 2^V$ such that there is a Player i vertex of $\mathcal{Z}(\mathcal{F})$ which has two successors. Then there is a Muller game $\mathcal{G} = (\mathcal{A}, \textsc{muller}(\mathcal{F}))$ with vertex set V where Player i has a winning strategy from some $v \in V$, but no positional one.

j) For every $\mathcal{F}_n$ with $n \in \mathbb{N}^+$ defined as in the game DJW_n we have that $\mathcal{Z}(\mathcal{F}_n)$ has at least $n!$ many leaves.