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Exercise Sheet 13

Infinite Games

Deadline: 28 Jan. 2014

Exercise 13.1 - Emptiness Game (3)

Complete the proof of Theorem 5.2 in the lecture notes. To this end, show that if $q_I \in W_0(\mathcal{G}(\mathscr{A}))$, then $\mathcal{L}(\mathscr{A}) \neq \emptyset$.

Exercise 13.2 - Emptiness Game, Example (4 + 1 + 1)

Consider the parity tree automaton $\mathscr{A} = (\{q_0, q_1, q_2, q_3, q_4\}, \{a, b\}, q_0, \Delta, \Omega)$ where Δ and Ω are defined by $\Omega(q_3) = 1$, $\Omega(q_0) = \Omega(q_4) = 2$, $\Omega(q_1) = \Omega(q_2) = 3$ and

$$\Delta = \{ (q_0, a, q_0, q_1),\ (q_0, b, q_0, q_3),\ (q_1, a, q_3, q_0),\ (q_1, b, q_0, q_2),\ (q_2, a, q_0, q_3),\ (q_2, b, q_0, q_1),\ (q_2, b, q_0, q_4),\ (q_3, a, q_0, q_3),\ (q_3, b, q_0, q_3),\ (q_4, a, q_0, q_1),\ (q_4, b, q_3, q_0) \}.$$

a) Construct the emptiness game $\mathcal{G}(\mathscr{A})$ and determine the winner from (ε, q_0).

b) Give a tree $t \in \mathcal{L}(\mathscr{A})$ as function $t\colon \mathbb{B}^* \to \{a, b\}$.

c) Give a precise description of $\mathcal{L}(\mathscr{A})$ using natural language.

Exercise 13.3 - Regular Trees (4 + 2)

A tree $t\colon \mathbb{B}^* \to \Sigma$ is regular iff it has finitely many different sub-trees, i.e., if the set $\{ t_w \mid w \in \mathbb{B}^* \}$ is finite.

Let $\mathscr{W} = (Q, \mathbb{B}, q_I, \delta, \lambda)$ be a deterministic finite word automaton (DFA) where we have replaced the set of accepting states by a labeling $\lambda\colon Q \to \Sigma$. We denote the unique state in which the run of $\mathscr{W}$ on $w \in \mathbb{B}^*$ is ending by $\delta^*(w)$. The automaton generates the tree $t_{\mathscr{W}}\colon \mathbb{B}^* \to \Sigma$ defined by $t_{\mathscr{W}}(w) = \lambda(\delta^*(w))$ for all $w \in \mathbb{B}^*$.

a) Show that a tree is regular if and only if it is generated by some DFA.

b) Show that every non-empty tree language recognized by a parity tree automaton contains a regular tree.

Exercise 13.4 - Challenge (2 Bonus Points)

In the lecture we proved that the emptiness problem for a parity tree automaton $\mathscr{A}$ is reducible to solving a parity game $\mathcal{G}(\mathscr{A})$, where $|\mathcal{G}(\mathscr{A})|$ is polynomial in $|\mathscr{A}|$.

Prove the converse, i.e., show that for every parity game $\mathcal{G}$ and vertex v of $\mathcal{G}$, there is a parity tree automaton $\mathscr{A}_{\mathcal{G},v}$ such that $v \in W_0(\mathcal{G})$ if and only if $\mathcal{L}(\mathscr{A}_{\mathcal{G},v}) \neq \emptyset$. Furthermore, $|\mathscr{A}_{\mathcal{G},v}|$ should be polynomial in $|\mathcal{G}|$.