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Problem Set 8

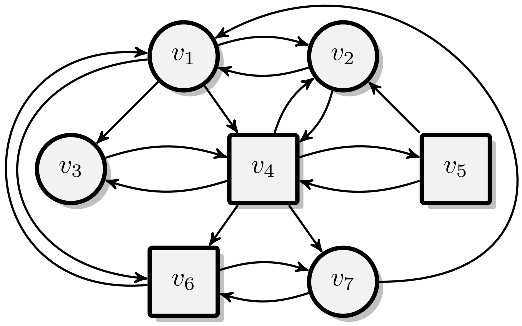
Infinite Games

Deadline: June, 13th 2016

Exercise 8.1 - Muller Games (3 Points)

Consider the Muller game $\mathcal{G}_1 = (\mathcal{A}_1, \textsc{Muller}(\mathcal{F}_1))$ with $\mathcal{A}_1$ as depicted below and

$$\mathcal{F}_1 = \{\{v_1, v_6\}, \{v_1, v_2, v_6, v_7\}, \{v_3, v_4\}, \{v_4, v_5\}\}.$$



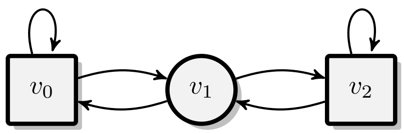
Determine the winning regions of $\mathcal{G}_1$ and uniform finite-state winning strategies for both players. Specify the strategies by giving a memory structure (not necessarily the same for both players) and a next-move function. You may use the graphical notation from the lecture.

Note: There was an update of the graphical notation since it was first presented. Refer to the lecture notes at `https://react.uni-saarland.de/teaching/infinite-game-16/lecture-notes.pdf` *for the most recent notation.*

Exercise 8.2 - LAR (3 Points)

Consider the Muller game $\mathcal{G}_2 = (\mathcal{A}_2, \textsc{Muller}(\mathcal{F}_2))$ with $\mathcal{A}_2$ as depicted below and

$$\mathcal{F}_2 = \{\{v_0\}, \{v_2\}, \{v_0, v_1, v_2\}\}.$$



Apply the LAR reduction to determine the winning regions of $\mathcal{G}_2$, where constructing the vertices reachable from $\{(v, \text{init}(v)) \mid v \in \{v_0, v_1, v_2\}\}$ suffices.

Exercise 8.3 - Büchi-Landweber Theorem (1 + 2 + 1 Points)

A (deterministic word) parity automaton $\mathscr{A} = (Q, \Sigma, q_I, \delta, \Omega)$ is a tuple consisting of

- a finite set Q of states,
- an alphabet Σ,
- an initial state $q_I \in Q$,
- a transition function $\delta\colon Q \times \Sigma \to Q$, and
- a coloring $\Omega\colon Q \to \mathbb{N}$.

The run $r = r_0 r_1 r_2 \cdots \in Q^\omega$ of $\mathscr{A}$ on an infinite input word $\alpha \in \Sigma^\omega$ is defined by $r_0 = q_I$ and $r_{n+1} = \delta(r_n, \alpha_n)$ for all $n \in \mathbb{N}$. This run is accepting if $\min(\mathrm{Inf}(\Omega(r_0)\Omega(r_1)\Omega(r_2)\cdots))$ is even. The language $\mathcal{L}(\mathscr{A})$ of $\mathscr{A}$ is the set of all input words whose run is accepting. A game $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ is ω-regular if there exists a parity automaton $\mathscr{A}$ with $\mathcal{L}(\mathscr{A}) = \mathrm{Win}$.

Prove the following statements formally:

a) Parity games are ω-regular.

b) Muller games are ω-regular.

c) Prove the Büchi-Landweber Theorem: Every ω-regular game is determined with uniform finite-state winning strategies.

   Hint: Construct a reduction.

Exercise 8.4 - Union-closed Muller Conditions (5 + 1 Points)

A family $\mathcal{F} \subseteq 2^V$ of sets is union-closed, if $F \cup F' \in \mathcal{F}$ for all $F, F' \in \mathcal{F}$. The family $\mathcal{F}$ is doubly union-closed, if $\mathcal{F}$ and $2^V \setminus \mathcal{F}$ are union-closed.

Show that doubly union-closed Muller conditions are equivalent to parity conditions, i.e.

a) Show that for every doubly union-closed $\mathcal{F} \subseteq 2^V$ there exists a coloring $\Omega\colon V \to \mathbb{N}$ with $\textsc{Muller}(\mathcal{F}) = \textsc{Parity}(\Omega)$. To this end, proceed as follows:

   – First show that the Zielonka tree encoding $\mathcal{F}$ is a path, i.e., that each vertex has at most one successor. (2 points)

   – Call the root of the Zielonka tree F_0 and, for each vertex labeled with F_i, call its unique child F_{i+1}. Construct a coloring Ω such that for each i and for each pair of vertices $v, v' \in F_i \setminus F_{i+1}$ we have $\Omega(v) = \Omega(v')$ and such that $\textsc{Muller}(\mathcal{F}) = \textsc{Parity}(\Omega)$ holds true. Describe your idea. (1 point)

   – Show that coloring constructed in the previous subtask actually has the stated property, i.e., show formally that $\textsc{Muller}(\mathcal{F}) = \textsc{Parity}(\Omega)$ holds true. (2 points)

b) Show that for every coloring $\Omega\colon V \to \mathbb{N}$ there exists a doubly union-closed $\mathcal{F} \subseteq 2^V$ such that $\textsc{Parity}(\Omega) = \textsc{Muller}(\mathcal{F})$ holds true. Show formally that the family $\mathcal{F}$ you construct is doubly union-closed.