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Problem Set 9

# Infinite Games

Deadline: June, 20th 2016

## Exercise 9.1 - Closure Properties (2 Points)

Show that $\mathbf{\Sigma}_n$ and $\mathbf{\Pi}_n$ are closed under union and intersection for every $n > 0$.

## Exercise 9.2 - Winning Conditions in the Borel Hierarchy (2 + 1 + 2 Points)

Let $V$ be a finite set. Prove each membership in the Borel hierarchy stated below.

1. $\text{wMuller}(\mathcal{F}) = \{\rho \in V^\omega \mid \text{Occ}(\rho) \in \mathcal{F}\} \in \mathbf{\Sigma}_2 \cap \mathbf{\Pi}_2$ for $\mathcal{F} \subseteq 2^V$.
2. $\text{coBüchi}(C) = \{\rho \in V^\omega \mid \text{Inf}(\rho) \subseteq C\} \in \mathbf{\Sigma}_2$ for $C \subseteq V$.
3. $\text{Parity}(\Omega) = \{\rho \in V^\omega \mid \min(\text{Inf}(\Omega(\rho_0)\Omega(\rho_1)\Omega(\rho_2)\cdots)) \text{ is even}\} \in \mathbf{\Sigma}_3 \cap \mathbf{\Pi}_3$ for $\Omega\colon V \to \mathbb{N}$.

*Hint*: Use the closure properties proven in Exercise 9.1.

## Exercise 9.3 - Wadge Games (2 + 2 + 2 Points)

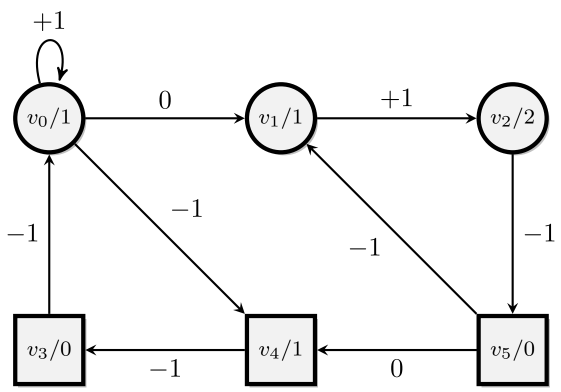
Fix $\mathbb{B} = \{0, 1\}$. A language $L \subseteq \mathbb{B}^\omega$ is *complete* for a level $\mathbf{\Sigma}_n$ of the Borel hierarchy over $\mathbb{B}$ if $L \in \mathbf{\Sigma}_n$ and $L' \leq L$ for every $L' \subseteq \mathbb{B}^\omega$ with $L' \in \mathbf{\Sigma}_n$. Completeness for $\mathbf{\Pi}_n$ is defined similarly.

1. Show that $0^*1(0+1)^\omega$ is complete for $\mathbf{\Sigma}_1$.
2. Show that $(0^*1)^\omega$ is complete for $\mathbf{\Pi}_2$.
3. Show that $(0^*1)^\omega$ is not in $\mathbf{\Sigma}_1 \cup \mathbf{\Pi}_1$.

## Exercise 9.4 - Energy Parity Games (1 + 1 + 1 Points)

In this exercise, we consider another variant of parity games, so-called energy parity games. As is the case with traditional parity games, Player 0 must ensure that the minimal color visited infinitely often is even. In contrast to traditional parity games, however, Player 0 has some fixed amount of initial energy, which she spends and regains when traversing edges. In addition to the parity condition, she also has to ensure that the energy level is always non-negative.

a) Read Section 2 of [1].[1] Compute $\text{EL}(w, v_0v_4v_3v_0v_0v_0)$ and $\text{EL}(w, v_1v_2c_5v_1v_2c_5)$, where $w$ is depicted as edge-labelling of the arena $\mathcal{A}$ below:




[1]The work [1] is available from the UdS-network from Elsevier via ScienceDirect. Make sure you obtain the journal version we reference, not the conference version. If you are unable to obtain a copy, please send a mail to `infinitegames16@react.uni-saarland.de`.

b) Determine the vertices $v$ in $\mathcal{G}$ for which there exists some initial credit $c_0 \in \mathbb{N}$ such that Player 0 wins the game $(\mathcal{A}, \textsc{PosEnergy}_{\mathcal{G}}(c_0) \cap \textsc{Parity}(\Omega))$ from $v$, where $\Omega$ is the coloring depicted in $\mathcal{A}$ above. For each such vertex compute the minimal initial credit that allows Player 0 to win.

c) At `http://react-teach.cs.uni-saarland.de` you will find a new problem which asks you to construct an energy parity game and specify the winning regions. Solve this problem using between seven and eight vertices, at most three colors, and the weights $-1$, $0$ and $+1$. Moreover, the winning regions for both players must not be empty and they must be different from the winning regions of your game if it is interpreted as a parity game.

Again, your constructed energy parity game will be given to other students to be solved. You will receive bonus points the harder your problem is for other students to solve. Moreover, you will only receive bonus points if you were able to solve your own game correctly.

# References


[1] K. Chatterjee, L. Doyen, *Energy Parity Games.* Theoretical Computer Science 458, pp. 49 - 60, Elsevier (2012).