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Infinite Games

Problem Set 10

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Exercise 10.1 - 16 for the price of 15 (16 Points)

1. Solving Muller games is in NP ∩ Co-NP if $\mathcal{F}$ is encoded

☐ by a circuit	☐ by a coloring function	☐ by a tree	☐ by an important subset	☐ by a boolean formula

2. In which of the following games is W_0 a trap for Player 1?

☐ Reachability	☐ Request Response	☐ Parity	☐ Safety	☐ Büchi	☐ Generalized Reachability	☐ Weak Muller

3. In a game with n vertices, which of the following winning conditions has the largest memory requirements for Player 0? (for sufficiently large n)

☐ Weak Muller	☐ Weak Parity	☐ Büchi	☐ Muller	☐ Parity	☐ Generalized Reachability	☐ Safety

4. What is the lower bound on the size of a winning strat. for Pl. 0 in weak Muller games?

☐ $\mathcal{O}(\log\|\mathcal{F}\|)$	☐ $\mathcal{O}(\|V\|\cdot\|\mathcal{F}\|)$	☐ $\mathcal{O}(2^{2^{\|V\|}})$	☐ $2^{\mathcal{O}(\|V\|)}$	☐ $\mathcal{O}(\|V\|^2)$	☐ $\mathcal{O}(\log^*(\|V\|))$	☐ 4

5. Which of the following winning conditions are a parity condition?

☐ Safety	☐ ReqRes	☐ wMuller	☐ Büchi	☐ Muller($\mathcal{F}$) with $\mathcal{F}$ doubly union-closed	☐ Reach	☐ GenReach	☐ Muller

6. Which of the following games are known to be solvable in polynomial time?

☐ Muller with $\mathcal{F}$ as Zielonka tree	☐ Energy Parity	☐ Büchi	☐ Solitary Parity	☐ Weak Parity	☐ Generalized Reachability	☐ Parity

7. Which of the following statements hold true?

☐ Büchi(F) $\in \mathbf{\Sigma}_2$	☐ Parity(Ω) $\in \mathbf{\Sigma}_2$	☐ wMuller($\mathcal{F}$) $\in \mathbf{\Sigma}_2 \cap \mathbf{\Pi}_2$	☐ Safety(S) $\in \mathbf{\Sigma}_2 \cap \mathbf{\Pi}_2$	☐ ReqRes($(Q_j, P_j)_{j\in[k]}$) $\in \mathbf{\Sigma}_3$	☐ Büchi(F) $\in \mathbf{\Pi}_2$	☐ Muller($\mathcal{F}$) $\in \mathbf{\Pi}_2$	☐ Reach(R) $\in \mathbf{\Pi}_1$

8. Which of the following winning conditions are prefix independent?

☐ wMuller	☐ coBüchi	☐ Büchi	☐ ReqRes	☐ Muller	☐ Parity	☐ Safety

9. For which of the following games does Player 1 have positional winning strategies?

☐ Muller	☐ Büchi	☐ Reachability	☐ co-Büchi	☐ Weak Parity	☐ Safety	☐ Parity	☐ Generalized Reachability

10. In which of the following games can Player 0 win with a uniform strategy?

☐ Muller	☐ Reachability	☐ Generalized Reachability	☐ co-Büchi	☐ Weak Parity	☐ Request-Response	☐ Parity

11. Let $\mathcal{G} = (\mathcal{A}, \text{Parity}(\Omega))$ be a parity game with even maximal color. How large is $|\text{Sh}(\mathcal{G})|$?

☐ $\|V\|$	☐ $2^{\|V\|}$	☐ $\|V\|^2$	☐ $1+\prod_{c\in\Omega(V),\,\mathrm{Par}(c)=1}\|\Omega^{-1}(c)\|$	☐ $\frac{\|V\|}{2}!+1$	☐ 42	☐ $\|V\|\cdot\|E\|$

12. Which of the following games are self-dual?

☐ co-Büchi	☐ Request-Response	☐ Reachability	☐ Parity	☐ Generalized Reachability	☐ Weak Muller	☐ Weak Parity

13. In which of the following games may Player 0 need memory?

☐ Büchi	☐ Weak Parity	☐ Safety	☐ Weak Muller	☐ Parity	☐ Request-Response	☐ Muller

14. Using game reductions, is it possible to reduce

☐ co-Büchi to Reachability?	☐ Büchi to Safety?	☐ Safety to Reachability?	☐ Parity to Muller?	☐ Muller to Büchi?	☐ Request-Response to Parity?	☐ Büchi to Parity?

15. Which of the following games are determined?

☐ Muller	☐ Safety	☐ Request-Response	☐ Parity	☐ Büchi	☐ Reachability	☐ co-Büchi

Exercise 10.2 - David, Wiesław, Robert and Michael walk into a bar...[1] (10 Bonus Points)

Given a family $\mathcal{F} \subseteq 2^V$ of subsets of a finite set V, recall that we defined its Zielonka tree $\mathcal{Z}(\mathcal{F})$ recursively as follows:

- The root of $\mathcal{Z}(\mathcal{F})$ is labeled by the set of all vertices.
- Children of a node labeled with $F \in \mathcal{F}$ are the $\subseteq$-maximal subsets $F' \subseteq F$ with $F' \notin \mathcal{F}$.
- Children of a node labeled with $F \notin \mathcal{F}$ are the $\subseteq$-maximal subsets $F' \subseteq F$ with $F' \in \mathcal{F}$.

We already had an example of such a tree on page 47 of the lecture notes. We say that a vertex v of $\mathcal{Z}(\mathcal{F})$ is a Player 0 vertex if its label is in $\mathcal{F}$. Otherwise, we call it a Player 1 vertex

Given a family $(Q_j, P_j)_{j\in[k]}$ of subsets $Q_j, P_j \subseteq V$ with $k \in \mathbb{N}$ we define the Rabin winning condition by

$$\textsc{rabin}((Q_j, P_j)_{j\in[k]}) = \{\rho \in V^\omega \mid \exists j \in [k].\, \mathrm{Inf}(\rho) \cap Q_j \neq \emptyset \textbf{ and } \mathrm{Inf}(\rho) \cap P_j = \emptyset\}$$

and the Streett winning condition by

$$\textsc{streett}((Q_j, P_j)_{j\in[k]}) = \{\rho \in V^\omega \mid \forall j \in [k].\, \mathrm{Inf}(\rho) \cap Q_j \neq \emptyset \textbf{ implies } \mathrm{Inf}(\rho) \cap P_j \neq \emptyset\}$$

Given an arena $\mathcal{A} = (V, V_0, V_1, E)$ we then call the games $\mathcal{G}_r = (\mathcal{A}, \textsc{rabin}((Q_j, P_j)_{j\in[k]}))$ and $\mathcal{G}_s = (\mathcal{A}, \textsc{streett}((Q_j, P_j)_{j\in[k]}))$ a Rabin game and a Street game, respectively.

Prove the following statements:

a) For every family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ it holds true that

$$\textsc{rabin}((Q_j, P_j)_{j\in[k]}) = V^\omega \setminus \textsc{streett}((Q_j, P_j)_{j\in[k]}).$$

b) For every coloring function $\Omega\colon V \to \mathbb{N}$ there exists a family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ such that $\textsc{Parity}(\Omega) = \textsc{rabin}((Q_j, P_j)_{j\in[k]})$.

c) For every coloring function $\Omega\colon V \to \mathbb{N}$ there exists a family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ such that $\textsc{Parity}(\Omega) = \textsc{streett}((Q_j, P_j)_{j\in[k]})$.

d) For every family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ there is a set $\mathcal{F} \subseteq 2^V$ such that $\textsc{rabin}((Q_j, P_j)_{j\in[k]}) = \textsc{Muller}(\mathcal{F})$.

e) For every family $(Q_j, P_j)_{j\in[k]}$ with $j \in \mathbb{N}$ and $Q_j, P_j \subseteq V$ there is a set $\mathcal{F} \subseteq 2^V$ such that $\textsc{streett}((Q_j, P_j)_{j\in[k]}) = \textsc{Muller}(\mathcal{F})$.

f) Let $\mathcal{F} \subseteq 2^V$. Every Player 0 vertex of $\mathcal{Z}(\mathcal{F})$ has at most one successor if and only if $\textsc{Muller}(\mathcal{F}) = \textsc{rabin}((Q_j, P_j)_{j\in[k]})$ for some family $(Q_j, P_j)_{j\in[k]}$ with $Q_j, P_j \subseteq V$.

g) Let $\mathcal{F} \subseteq 2^V$. Every Player 1 vertex of $\mathcal{Z}(\mathcal{F})$ has at most one successor if and only if $\textsc{Muller}(\mathcal{F}) = \textsc{streett}((Q_j, P_j)_{j\in[k]})$ for some family $(Q_j, P_j)_{j\in[k]}$ with $Q_j, P_j \subseteq V$.

h) Let $\mathcal{F} \subseteq 2^V$. Every vertex of $\mathcal{Z}(\mathcal{F})$ has at most one successor if and only if $\textsc{Muller}(\mathcal{F}) = \textsc{Parity}(\Omega)$ for some coloring function $\Omega\colon V \to \mathbb{N}$.

i) Let $\mathcal{Z}(\mathcal{F})$ be the Zielonka tree for some $\mathcal{F} \subseteq 2^V$ such that there is a Player i vertex of $\mathcal{Z}(\mathcal{F})$ which has two successors whose labels have a nonempty intersection. Then there is a Muller game $\mathcal{G} = (\mathcal{A}, \textsc{Muller}(\mathcal{F}))$ with vertex set V where Player i has a winning strategy from some $v \in V$, but no positional one.

j) For every $\mathcal{F}_n$ with $n \in \mathbb{N}^+$ defined as in the game $\mathcal{DJW}_n$ we have that $\mathcal{Z}(\mathcal{F}_n)$ has at least $n!$ many leaves.

[1] They are followed by a countably infinite number of mathematicians. David orders a beer. Wiesław orders half a beer. Robert orders a quarter of a beer. The barkeeper interrupts them, pours two beers and says: "Know your limits."