First, we present a full proof of correctness for the construction presented in the lecture. Then, we do something even better: we present constructions due to Yannick for both directions of the proof that are simpler than the ones presented in the lecture and the first one even avoids the case distinction.

1 Construction from the Lecture

Recall that we want to show the following statement:

> For every recursive $g\colon \mathbb{N} \to \mathbb{N}$ with $\mathrm{dom}(g) \neq \emptyset$ there exists a total recursive $f\colon \mathbb{N} \to \mathbb{N}$ with $\mathrm{dom}(g) = f(\mathbb{N})$.

Also, only the case where $|\mathrm{dom}(g)| = \infty$ remains to be considered. We define $f'\colon \mathbb{N} \to \mathbb{N}$ via the scheme of primitive recursion as

$$f'(0) = \mu t\colon T_1 e[t]_1[t]_2 \quad \text{and} \quad f'(z+1) = \mu t\colon T_1 e[t]_1[t]_2 \wedge t > f'(z)\,,$$

where e is an index of g, i.e., we have $g(x) = U(\mu y\colon T_1 exy)$. Now, define $f(z) = [f'(z)]_1$. The function f is recursive by construction. Hence, it remains to show that it is total and that $f(\mathbb{N}) = \mathrm{dom}(g)$.

To this end, we use the following property of the paring function: we have $[x, y] \geq 2^x - 1$ for every x, independently of y. Thus, by picking x large enough, we can make $[x, y]$ arbitrarily large.

To show that f is total, it suffices to show that f' is total, as $[\cdot]_1$ is total. As $\mathrm{dom}(g) \neq \emptyset$, there is some $x \in \mathrm{dom}(g)$. Thus, by the KNFT there is also a y such that $T_1 exy$ holds. Hence, there is a $t = [x, y]$ such that $T_1 e[t]_1[t]_2$ holds, which implies that $f'(0)$ is defined.

Now, consider $f'(z + 1)$. We have to find for every possible value of $f'(z)$ a t with $t > f(z + 1)$ and $T_1 e[t]_1[t]_2$. As $\mathrm{dom}(g)$ is infinite, we can always pick a large enough $x \in \mathrm{dom}(g)$ (with an associated y with $T_1 exy$) such that $[x, y] > f'(z)$. Thus, $f'(z + 1)$ is defined.

To show that $f(\mathbb{N}) = \mathrm{dom}(g)$, we first argue $f(\mathbb{N}) \subseteq \mathrm{dom}(g)$: let $x \in f(\mathbb{N})$. Then, there is a y such that $[x, y] \in f'(\mathbb{N})$. Hence, as f' only returns numbers $[x, y]$ with $T_1 exy$, we conclude $x \in \mathrm{dom}(g)$ by the KNFT.

Now, we show $\mathrm{dom}(g) \subseteq f(\mathbb{N})$: let $x \in \mathrm{dom}(g)$, i.e., there is a y such that $T_1 exy$. Towards a contradiction assume that $x \notin f(\mathbb{N})$. Then, $[x, y] \notin f'(\mathbb{N})$.

It cannot be the case that $[x, y]$ is strictly smaller than $f(0)$, as $f'(0)$ is the smallest t with $T_1 e[t]_1[t]_2$. By assumption, $[x, y]$ is not equal to $f'(0)$. Also, $[x, y]$ cannot satisfy $f'(0) < [x, y] < f'(1)$, as as $f'(1)$ is the smallest $t > f'(0)$ with $T_1 e[t]_1[t]_2$. Repeating this argument, we have that $[x, y]$ is strictly greater than $f'(i)$ for every i. But the set $f'(\mathbb{N})$ is unbounded, as $\mathrm{dom}(g)$ is infinite. Hence, $[x, y]$ is strictly greater than every natural number, i.e., we have derived our contradiction.

2 Yannick's Constructions

- "$\Leftarrow$": given a recursive g define a total recursive f with $f(\mathbb{N}) = \mathrm{dom}(g)$.

  Let e be an index of g and $x_0 \in \mathrm{dom}(g) \neq \emptyset$. Now, define $f\colon \mathbb{N} \to \mathbb{N}$ by

  $$f(t) = \begin{cases} [t]_1 & \text{if } T_1 e [t]_1 [t]_2, \\ x_0 & \text{otherwise,} \end{cases}$$

  which is primitive recursive (as case distinction and T_1 are primitive recursive) and thus total. Furthermore, if $x \in \mathrm{dom}(g)$, then there is a y such that $T_1 exy$ holds, i.e., $f([x, y]) = x$. Thus, $x \in f(\mathbb{N})$.

  On the other hand, f only returns elements from $\mathrm{dom}(g)$: either x_0, or $[t]_1$ such that $T_1 e [t]_1 [t]_2$ holds, which implies that $[t]_1$ is in $\mathrm{dom}(g)$.

- "$\Rightarrow$": given a total recursive f define a recursive g with $\mathrm{dom}(g) = f(\mathbb{N})$.

  Let $g(x) = \mu y\colon f(y) = x$. As f is total, $g(x)$ is only undefined, if there is no y such that $f(y) = x$, i.e., if $x \notin f(\mathbb{N})$. On the other hand, if $g(x)$ is defined, then there is an y such that $f(y) = x$, i.e., if $x \in f(\mathbb{N})$. Thus, $\mathrm{dom}(g) = f(\mathbb{N})$.