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Summer term 2015
Problem Set 10
Due: Tuesday, July 7th, 2015

Recursion Theory

Problem 1: Recursive Inseparability — 4 Points

Two sets $A, B \subseteq \mathbb{N}$ are *recursively inseparable*, if there is no recursive set $C \subseteq \mathbb{N}$ such that $A \subseteq C$ and $B \subseteq \mathbb{N} \setminus C$. Let $K_0 = \{e \mid \varphi_e(e) = 0\}$ and $K_1 = \{e \mid \varphi_e(e) = 1\}$.

Show that K_0 and K_1 are enumerable and recursively inseparable.

Hint: Assume there is such a C and check whether an index of its characteristic function is in C or not.

Problem 2: Uniformization — 4 Points

Let $R, R_0 \subseteq \mathbb{N}^2$. We say that $R_0 \subseteq R$ uniformizes R, if

- R and R_0 have the same domain, i.e., $\{x \mid \exists y\, Rxy\} = \{x \mid \exists y\, R_0xy\}$, and
- R_0 is functional, i.e., for every $x \in \{x \mid \exists y\, R_0xy\}$ there is a unique y_0 with R_0xy_0.

Show: every enumerable $R \subseteq \mathbb{N}^2$ is uniformized by an enumerable $R_0 \subseteq \mathbb{N}^2$.

Recall: $R \subseteq \mathbb{N}^2$ is enumerable, if there exists a recursive $f\colon \mathbb{N}^2 \to \mathbb{N}$ with $\mathrm{dom}(f) = R$.

Problem 3: Fun with Functions, Part 2 — 4 Points

Show the existence of an infinite sequence $(m_i)_{i\in\mathbb{N}}$ of pairwise different natural numbers satisfying $W_{m_i} = \{m_{i+1}\}$ for every i, where $W_e = \mathrm{dom}(\varphi_e^{(1)})$. To this end, do the following:

a) Construct a total recursive function $f\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$

$$W_{f(x,y)} = \begin{cases} \{\varphi_x(\varphi_x(y))\} & \text{if } \varphi_x(y)\downarrow \text{ and } \varphi_x(\varphi_x(y))\downarrow, \\ \emptyset & \text{otherwise.} \end{cases}$$

b) Construct a total recursive function $g\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$: $W_{g(x,y)} = W_{f(x,y)}$ and $g(x, y) > y$.

c) Show that there exists an e such that φ_e is total and such that for all y: $W_{\varphi_e(y)} = \{\varphi_e(\varphi_e(y))\}$ and $\varphi_e(y) > y$.

d) Define the sequence $(m_i)_{i\in\mathbb{N}}$ using φ_e and prove that it has the desired properties.